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On modified Dunkl generalization of Szász-operators via – *q* calculus

M. Mursaleen and **Md. Nasiruzzaman**

Department of Mathematics, Aligarh Muslim University, Aligarh–202002, India
mursaleenm@gmail.com; nasir3489@gmail.com

Abstract: The purpose of this paper is to introduce a modification of sequence of Dunkl generalization of exponential functions via q -calculus which is based on a continuously differentiable function τ on $[0,\infty)$. Uniform approximation by such a sequence has been studied and degree of approximation by the operators has been obtained. Moreover, We obtain some approximation results via well known Korovkin's type theorem, weighted Korovkin's type theorem convergence properties by using the modulus of continuity and the rate of convergence of the operators for functions belonging to the Lipschitz class is presented.
Keywords and phrases: q -integers; Dunkl analogue; Szász operator; q - Szász-Mirakjan-Kantrovich; modulus of continuity; Peetre's K-functional.
AMS Subject Classification (2010): 41A25, 41A36, 33C45.

1. Introduction and preliminaries

In 1912, S.N Bernstein [3] introduced the following sequence of operators $B_n : C[0,1] \to C[0,1]$ defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \qquad x \in 0,1] \tag{1.1}$$

for $n \in \mathbb{N}$ and $f \in C[0,1]$.

In 1950, for $x \geq 0$, Szász [27] introduced the operators

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \qquad f \in C[0,\infty). \tag{1.2}$$

In the field of approximation theory, the application of q -calculus emerged as a new area in the field of approximation theory. The first q -analogue of the well-known Bernstein polynomials was introduced by Lupaș by applying the idea of q -integers [12]. In 1997 Phillips [34] considered another q -analogue of the classical Bernstein polynomials. Later on, many authors introduced q -generalizations of various operators and investigated several approximation properties [13,14,15,16].

We now present some basic definitions and concept details of the q -calculus which are used in this paper.

Definition 1.1. For $|q|<1,$ the basic (or q-) number $[\lambda]_q$ is defined by

$$[\lambda]_q = \begin{cases} \dfrac{1-q^{\lambda}}{1-q} & (\lambda \in \mathbb{C}) \\ \sum_{k=0}^{n-1} q^k = 1+q+q^2+\cdots+q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases} \tag{1.3}$$

Definition 1.2. For $|q|<1,$ the basic (or q-) the q-factorial $[n]_q!$ is defined by

$$[n]_q! = \begin{cases} 1 & (n=0) \\ \prod_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases} \tag{1.4}$$

Definition 1.3. For $|q|<1,$ the generalized basic (or q-) binomial coefficient $\begin{bmatrix} \lambda \\ n \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} \lambda \\ n \end{bmatrix}_q = \frac{(q^{-\lambda};q)_n}{(q;q)_n} (-q^{\lambda})^n q^{-\binom{n}{2}} \qquad (\lambda \in \mathbb{C}; n \in \mathbb{N}_0). \tag{1.5}$$

For $q, \lambda, \nu \in \mathbb{C}$ $(|q|<1)$, the basic (or q-) shifted factorial $(\lambda;q)_\nu$ is defined by (see, for example, [19], [21] and [22]; see also the recent works [20,23] dealing with the q-analysis)

$$(\lambda;q)_\nu = \prod_{j=0}^{\infty} \left(\frac{1-\lambda q^j}{1-\lambda q^{\nu+j}} \right) \qquad (|q|<1;\ \lambda, \nu \in \mathbb{C}), \tag{1.6}$$

so that

$$(\lambda;q)_n := \begin{cases} 1 & (n=0) \\ \prod_{j=0}^{n-1} (1-\lambda q^j) & (n \in \mathbb{N}) \end{cases} \tag{1.7}$$

and

$$(\lambda;q)_\infty := \prod_{j=0}^{\infty} (1-\lambda q^j) \qquad (|q|<1;\ \lambda \in \mathbb{C}), \tag{1.8}$$

where, as usual, $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{N}$ denotes the set of positive integers *with*

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0,1,2,\cdots\}.$$

For convenience, we write

$$(a_1,\cdots,a_r;q)_n = (a_1;q)_n \cdots (a_r;q)_n \tag{1.9}$$

and

$$(a_1,\cdots,a_r;q)_\infty = (a_1;q)_\infty \cdots (a_r;q)_\infty. \tag{1.10}$$

In our investigation, we shall make use of the basic (or q-) hypergeometric function ${}_r\Phi_s$ with r numerator and s denominator parameters, which is defined by (see, for example, [22, p. 347, Eq. 9.4 (272)])

$$ {}_r\Phi_s\begin{bmatrix} a_1,\cdots,a_r; & b_1,\cdots,b_s; & \\ & & q,z \\ b_1,\cdots,b_s; & & \end{bmatrix} := \sum_{k=0}^{\infty}(-1)^{(1-r+s)k}\, q^{(1-r+s)\binom{k}{2}} \cdot \frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_s;q)_k}\frac{z^k}{(q;q)_k}, \tag{1.11}$$

provided that the *generalized basic* (or q-) *hypergeometric series* in [22] converges.

Theorem 1. (The q-Binomial Theorem) *For $|q|<1$, the basic (or q-) binomial theorem is given by*

$$ {}_1\Phi_0\begin{bmatrix} \lambda; & \overline{\quad}; & \\ & & q,z \\ \underline{\quad}; & & \end{bmatrix} := \sum_{k=0}^{\infty}\frac{(\lambda;q)_k}{(q;q)_k}z^k = \frac{(\lambda z;q)_\infty}{(z;q)_\infty} \qquad (|q|<1;|z|<1), \tag{1.12}$$

Remark 1. The basic (or q-) binomial theorem (1.12) (also known as *Heine's Theorem*) simplifies considerably to the following form when we set $\lambda = q^{-n}$ $(n \in \mathbb{N}_0)$:

$$ {}_1\Phi_0\begin{bmatrix} q^{-n}; & \overline{\quad}; & \\ & & q,z \\ \underline{\quad}; & & \end{bmatrix} := \sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}z^k = (zq^{-n};q)_n$$

$$= \left(\frac{q}{z}\right)_n\left(-\frac{z}{q}\right)^n q^{-\binom{n}{2}} \qquad (|q|<1; n\in\mathbb{N}_0).$$

Definition 1.4. For $|q|<1$, the basic (or q-) exponential function $e_q(z)$ of the first kind is defined by

$$e_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} =: {}_1\Phi_0\left[\begin{matrix} 0; & \overline{\quad}; & \\ & & q, z \\ \overline{\quad}; & & \end{matrix}\right] = \frac{1}{(z;q)_\infty}, \tag{1.13}$$

where we have used the *special* case of the q-binomial theorem (1.12) when $\lambda = 0$.

Definition 1.5. For $|q|<1$, the basic (or q-) exponential function $E_q(z)$ of the second kind is defined by

$$E_q(z) := \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{z^k}{(q;q)_k} =: {}_0\Phi_0\left[\begin{matrix} \overline{\quad}; & \overline{\quad}; & \\ & & q, -z \\ \overline{\quad}; & & \end{matrix}\right] = (-z;q)_\infty, \tag{1.14}$$

where we have used the *limit* case of the q-binomial theorem (1.12) when z is replaced by $\frac{z}{\lambda}$ and $\lambda \to \infty$.

Remark 2. It is easily seen by applying the definitions (1.13) and (1.14) that

$$\lim_{q\to 1}\{e_q((1-q)z)\} = e^z = \lim_{q\to 1}\{E_q((1-q)z)\} \quad \text{and} \quad e_q(z)\cdot E_q(-z) = 1. \tag{1.15}$$

Our investigation is to construct a linear positive operators generated by generalization of exponential function for defined by [20]

$$e_\mu(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_\mu(n)}.$$

Here

$$\gamma_\mu(2k) = \frac{2^{2k} k!\,\Gamma\left(k+\mu+\frac{1}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)},$$

and

$$\gamma_\mu(2k+1) = \frac{2^{2k+1} k!\,\Gamma\left(k+\mu+\frac{3}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)}.$$

The recursion formula for γ_μ is given by

$$\gamma_\mu(k+1) = (k+1+2\mu\theta_{k+1})\gamma_\mu(k), \quad k = 0,1,2,\cdots,$$

where $\mu > -\frac{1}{2}$ and

$$\theta_k = \begin{cases} 0 & \text{if} k \in 2\mathbb{N} \\ 1 & \text{if} k \in 2\mathbb{N}+1. \end{cases}$$

Sucu defined a Dunkl analogue of Szász operators via a generalization of the exponential function [20] as follows:

$$S_n^*(f;x) := \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} f\left(\frac{k+2\mu\theta_k}{n}\right), \tag{1.16}$$

where $x \geq 0, \quad f \in C[0,\infty), \mu \geq 0, \quad n \in \mathbb{N}$.

Cheikh et al., stated the q-Dunkl classical q-Hermite type polynomials and gave definitions of q-Dunkl analogues of exponential functions and recursion relations for $\mu > -\frac{1}{2}$ and $0 < q < 1$.

$$e_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_{\mu,q}(n)}, \quad x \in 0,\infty) \tag{1.17}$$

$$E_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{\gamma_{\mu,q}(n)}, \quad x \in 0,\infty) \tag{1.18}$$

$$\gamma_{\mu,q}(n+1) = \left(\frac{1-q^{2\mu\theta_{n+1}+n+1}}{1-q}\right)\gamma_{\mu,q}(n), \quad n \in \mathbb{N}, \tag{1.19}$$

$$\theta_n = \begin{cases} 0 & \text{if} n \in 2\mathbb{N}, \\ 1 & \text{if} n \in 2\mathbb{N}+1. \end{cases}$$

An explicit formula for $\gamma_{\mu,q}(n)$ is

$$\gamma_{\mu,q}(n) = \frac{(q^{2\mu+1},q^2)_{[\frac{n+1}{2}]}(q^2,q^2)_{[\frac{n}{2}]}}{(1-q)^n}\gamma_{\mu,q}(n), \quad n \in \mathbb{N}.$$

And some of the special cases of $\gamma_{\mu,q}(n)$ are defined as:

$$\gamma_{\mu,q}(0) = 1, \quad \gamma_{\mu,q}(1) = \frac{1-q^{2\mu+1}}{1-q}, \quad \gamma_{\mu,q}(2) = \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right),$$

$$\gamma_{\mu,q}(3) = \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^{2\mu+3}}{1-q}\right),$$

$$\gamma_{\mu,q}(4) = \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^{2\mu+3}}{1-q}\right)\left(\frac{1-q^4}{1-q}\right).$$

Gürhan Içöz gave the Dunkl generalization of Szász operators via q-calculus as:

$$D_{n,q}(f;x)=\frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} f\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right), \tag{1.20}$$

for $\mu>\frac{1}{2}$, $x\geq 0$, $0<q<1$ and $f\in C[0,\infty)$.

Previous studies demonstrate that providing a better error estimation for positive linear operators plays an important role in approximation theory, which allows us to approximate much faster to the function being approximated. In [5, 6], various better approximation properties of the Szász-Mirakjan-Kantrovich operators, and q-Szász-Mirakjan-Kantrovich operators, were investigated.

Motivated essentially by by Gürhan Içöz, the recent investigation of Dunkl generalization of Szász-Mirakjan operators via q-calculus the Uniform approximation by such a sequence has been studied and degree of approximation by the operators has been obtained which is based on a continuously differentiable function τ on $[0,\infty)$ by $\tau(0)=0$ and $\inf_{x\in\mathbb{R}^+}\tau'(x)\geq 1$. We have showed that our modified operators give a degree of approximation by. We have proved several approximation results. Several other related results have also been considered.

2. Construction of operators and moments estimation

We modify the q Dunkl analogue of Szász-operators by [2].

Let $\tau(x)$ be a continuously differentiable functions defined on $\mathbb{R}^+$ satisfying

1. $\tau(0)=0$,
2. $\inf_{x\in\mathbb{R}^+}\tau'(x)\geq 1$.

Then for any $0<q<1$, $\mu>\frac{1}{2n}$ and $n\in\mathbb{N}$ we define

$$\begin{aligned} L_{n,q}^{\tau}(f;x)&=\frac{1}{e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\frac{([n]_q\tau(x))^k}{\gamma_{\mu,q}(k)}(fo\tau^{-1})\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right)\\ &=(D_{n,q}(fo\tau^{-1})o\tau)(x)\\ &=\frac{1}{e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\frac{([n]_q\tau(x))^k}{\gamma_{\mu,q}(k)} f\left(\tau^{-1}\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right)\right) \end{aligned} \tag{2.1}$$

where $e_{\mu,q}(x)$, $\gamma_{\mu,q}$ are defined in (1.17),(1.19) and $f\in C_\zeta[0,\infty)$ with $\zeta\geq 0$ and

$$C_\zeta[0,\infty)=\{f\in C[0,\infty):|f(t)|\leq M(1+t)^\zeta,\quad for\quad some\quad M>0,\ \zeta>0\}. \tag{2.2}$$

Note that the function $\tau(x)=x+x^2$ satisfies the conditions 1 and 2. If $\tau=t$, then $L_{n,q}^{\tau}=D_{n,q}$. It is easily seen that

Lemma 2.1. Let $L_{n,q}^{\tau}(.;.)$ be the operators given by (2.1). Then for each continuously differentiable function τ on R^{+}, we have we have the following identities*:*

1. $L_{n,q}^{\tau}(1;x)=1$,

2. $L_{n,q}^{\tau}(\tau;x)=\tau(x)$,

3. $\tau(x)^2+\left(q^{2\mu}[1-2\mu]_q\dfrac{e_{\mu,q}(q[n]_q\tau(x))}{e_{\mu,q}([n]_q\tau(x))}\right)\dfrac{\tau(x)}{[n]_q}\leq L_{n,q}^{\tau}(\tau^2;x)\leq\tau(x)^2+\left([1+2\mu]_q\right)\dfrac{\tau(x)}{[n]_q}$

Proof.

1. $L_{n,q}^{\tau}(1;x)=\dfrac{1}{e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\dfrac{([n]_q\tau(x))^k}{\gamma_\mu(k)}=1$.

2. $$L_{n,q}^{\tau}(\tau;x)=\frac{1}{e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\frac{([n]_q\tau(x))^k}{\gamma_\mu(k)}\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right)$$
$$=\frac{1}{[n]_q e_{\mu,q}([n]_q\tau(x))}\sum_{k=1}^{\infty}\frac{([n]_q\tau(x))^k}{\gamma_\mu(k-1)}$$
$$=\tau(x)$$

3. $$L_{n,q}^{\tau}(\tau^2;x)=\frac{1}{e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\frac{([n]_q\tau(x))^k}{\gamma_\mu(k)}\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right)^2$$
$$=\frac{1}{[n]_q^2 e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\frac{([n]_q\tau(x))^k}{\gamma_\mu(k-1)}\left(\frac{1-q^{2\mu\theta_k+k}}{1-q}\right)$$
$$=\frac{1}{[n]_q^2 e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\frac{([n]_q\tau(x))^{k+1}}{\gamma_\mu(k)}\left(\frac{1-q^{2\mu\theta_{k+1}+k+1}}{1-q}\right).$$

From [16] we know that

$$[2\mu\theta_{k+1}+k+1]_q=[2\mu\theta_k+k]_q+q^{2\mu\theta_k+k}[2\mu(-1)^k+1]_q, \tag{2.3}$$

Now by separating to the even and odd terms and using (2.3), we get

$$L_{n,q}^{\tau}(\tau^2;x)=\frac{1}{[n]_q^2 e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\frac{([n]_q\tau(x))^{k+1}}{\gamma_\mu(k)}\left(\frac{1-q^{2\mu\theta_{k+1}+k+1}}{1-q}\right)$$

$$+\frac{[1+2\mu]_q}{[n]_q^2 e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\frac{([n]_q\tau(x))^{2k+1}}{\gamma_\mu(2k)}q^{2\mu\theta_{2k}+2k}$$
$$+\frac{[1-2\mu]_q}{[n]_q^2 e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\frac{([n]_q\tau(x))^{2k+2}}{\gamma_\mu(2k)}q^{2\mu\theta_{2k+1}+2k+1}.$$

We know the inequality

$$[1-2\mu]_q \leq [1+2\mu]_q. \tag{2.4}$$

Therefore by using (2.4) we have

$$L_{n,q}^{\tau}(\tau^2;x) \geq (\tau(x))^2 + \frac{\tau(x)[1-2\mu]_q}{[n]_q e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\frac{(q[n]_q\tau(x))^{2k}}{\gamma_\mu(2k)}$$
$$+\frac{q^{2\mu}\tau(x)[1-2\mu]_q}{[n]_q e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\frac{(q[n]_q\tau(x))^{2k+1}}{\gamma_\mu(2k+1)}$$
$$\geq (\tau(x))^2 + q^{2\mu}[1-2\mu]_q\frac{e_{\mu,q}(q[n]_q\tau(x))}{e_{\mu,q}([n]_q\tau(x))}\frac{\tau(x)}{[n]_q}.$$

Similarly on the other hand we have

$$L_{n,q}^{\tau}(\tau^2;x) \leq (\tau(x))^2 + [1+2\mu]_q\frac{\tau(x)}{[n]_q}.$$

Which completes the proof.

Lemma 2.2. Let the operators $L_{n,q}^{\tau}(.;.)$ be given by (2.1). Then for each continuously differentiable function $\tau(x)$ on $\mathbb{R}^+$, we have

1. $L_{n,q}^{\tau}(\tau-\tau(x);x)=0$,
2. $L_{n,q}^{\tau}((\tau-\tau(x))^2;x) \leq [1+2\mu]_q\frac{\tau(x)}{[n]_q}$.

3. Main results

We obtain the Korovkin's type approximation properties for our operators defined by (2.1). The Korovkin theory is an important area of study in approximation theory; see [3]. Briefly speaking, the Korovkin theorem says that if a sequence of linear positive operators approximates uniformly the test functions $1,t$ and t^2, then it approximates all continuous functions defined on a bounded interval. This theorem was extended to unbounded intervals and a weighted Korovkin type theorem in a subspace of continuous functions on the real axis $\mathbb{R}$ was proved in [8, 9]. Also it was shown that these test functions can be replaced by $1,\tau,\tau^2$ under certain additional conditions on τ (see Theorem 3.3). We show here that the operator defined in (2.1) satisfies the required conditions for the weighted Korovkin type theorem.

Let $C_B(\mathbb{R}^+)$ be the set of all bounded and continuous functions on $\mathbb{R}^+=[0,\infty)$, which is linear normed space with

$$\| f \|_{C_B} = \sup_{x\geq 0} | f(x) | .$$

Let

$$H := \{ f : x \in 0,\infty), \frac{f(x)}{1+\tau(x)^2} \quad is \quad convergent \quad as \quad x \to \infty \}.$$

Theorem 3.1. *Let* $L_{n,q}^{\tau}(.;.)$ *be the operators defined by (2.1). Then for any function* $f \in C_{\zeta}[0,\infty) \cap H$, $\zeta \geq 2$,

$$\lim_{n\to\infty} L_{n,q}^{\tau}(f;x) = f(x)$$

is uniformly on each compact subset of $[0,\infty)$, where $x \in [\frac{1}{2}, b)$, $b > \frac{1}{2}$.

Proof. The proof is based on Lemma 2.1 and well known Korovkin's theorem regarding the convergence of a sequence of linear and positive operators, so it is enough to prove the conditions

$$\lim_{n\to\infty} L_{n,q}^{\tau}(\tau^j; x) = \tau(x)^j, \quad j = 0,1,2, \quad \{as\ n \to \infty\}$$

uniformly on $[0,1]$.

Clearly $\frac{1}{[n]_q} \to 0 \ (n \to \infty)$ we have

$$\lim_{n\to\infty} L_{n,q}^{\tau}(\tau; x) = \tau(x), \quad \lim_{n\to\infty} L_{n,q}^{\tau}(\tau^2; x) = \tau(x)^2.$$

Which complete the proof.

We recall the weighted spaces of the functions on $\mathbb{R}^+$, which are defined as follows:

$$P_{\rho}(\mathbb{R}^+) = \left\{ f : | f(x) | \leq M_f \rho(x) \right\},$$

$$Q_{\rho}(\mathbb{R}^+) = \left\{ f : f \in P_{\rho}(\mathbb{R}^+) \cap C[0,\infty) \right\},$$

$$Q_{\rho}^{k}(\mathbb{R}^+) = \left\{ f : f \in Q_{\rho}(\mathbb{R}^+) \quad and \quad \lim_{x\to\infty} \frac{f(x)}{\rho(x)} = k(k \quad is \quad a \quad constant) \right\},$$

where $\rho(x) = 1 + \tau(x)^2$ is a weight function and M_f is a constant depending only on f.

Note that $Q_{\rho}(\mathbb{R}^+)$ is a normed space with the norm $\| f \|_{\rho} = \sup_{x\geq 0} \frac{| f(x) |}{\rho(x)}$.

Lemma 3.2. ([7]) The linear positive operators L_n, $n \geq 1$ act from $Q_{\rho}(\mathbb{R}^+) \to P_{\rho}(\mathbb{R}^+)$ if and only if

$$\| L_n(\varphi; x) \| \leq K\varphi(x),$$

where $\varphi(x) = 1 + x^2$, $x \in \mathbb{R}^+$ and K is a positive constant.

Theorem 3.3. *([7]) Let* $\{L_n\}_{n\geq 1}$ *be a sequence of positive linear operators acting from* $Q_\rho(\mathbb{R}^+) \to P_\rho(\mathbb{R}^+)$ *and satisfying the condition*

$$\lim_{n\to\infty} \| L_n(\rho^\nu) - \rho^\nu \|_\varphi = 0, \quad \nu = 0,1,2.$$

Then for any function $f \in Q_\rho^k(\mathbb{R}^+)$, we have

$$\lim_{n\to\infty} \| L_n(f;x) - f \|_\varphi = 0.$$

Theorem 3.4. *Let* $L_{n,q}^\tau(.;.)$ *be the operators defined by (2.1). Then for each function* $f \in Q_\rho^k(\mathbb{R}^+)$ *we have*

$$\lim_{n\to\infty} \| L_{n,q}^\tau(f;x) - f \|_\rho = 0.$$

Proof. From Lemma 2.1 and Theorem 3.3 for $\nu = 0$, the first condition is fulfilled. Therefore

$$\lim_{n\to\infty} \| L_{n,q}^\tau(1;x) - 1 \|_\rho = 0.$$

Similarly From Lemma 2.1 and Theorem 3.3 for $\nu = 1,2$ we have that

$$\sup_{x\in[0,\infty)} \frac{| L_{n,q}^\tau(\tau;x) - \tau(x) |}{1+\tau(x)^2} \leq \frac{1}{2[n]_q} \sup_{x\in[0,\infty)} \frac{1}{1+\tau(x)^2}$$
$$= \frac{1}{2[n]_q},$$

which imply that

$$\lim_{n\to\infty} \| L_{n,q}^\tau(\tau;x) - \tau(x) \|_\rho = 0.$$

$$\sup_{x\in[0,\infty)} \frac{| L_{n,q}^\tau(\tau^2;x) - \tau(x)^2 |}{1+\tau(x)^2} \leq \frac{|[1+2\mu]_q - 1|}{[n]_q} \sup_{x\in[0,\infty)} \frac{\tau(x)}{1+\tau(x)^2}$$
$$+ \frac{1}{4[n]_q^2} |[1+2\mu]_q - 1| \sup_{x\in[0,\infty)} \frac{1}{1+\tau(x)^2}$$

which imply that

$$\lim_{n\to\infty} \| L_{n,q}^\tau(\tau^2;x) - \tau(x)^2 \|_\rho = 0.$$

This complete the proof.

4. Rate of Convergence

Here we calculate the rate of convergence of operators (2.1) by means of modulus of continuity and Lipschitz type maximal functions.

Let $f \in C_B[0,\infty]$, the space of all bounded and continuous functions on $[0,\infty)$, $\tau(x)$ is a continuously differentiable function on $\mathbb{R}^+$ and $x \geq \frac{1}{2n}$, $n \in \mathbb{N}$. Then for $\delta > 0$, the modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by

$$\omega(f,\delta) = \sup_{|\tau(t)-\tau(x)|\leq\delta} |f(t) - f(x)|, \quad t \in 0,\infty). \tag{4.1}$$

It is known that $\lim_{\delta\to 0+}\omega(f,\delta) = 0$ for $f \in C_B[0,\infty)$ and for any $\delta > 0$ one has

$$|f(t) - f(x)| \leq \left(\frac{|\tau(t)-\tau(x)|}{\delta} + 1\right)\omega(f,\delta). \tag{4.2}$$

Theorem 4.1. *Let* $L^{\tau}_{n,q}(.;.)$ *be the operators defined by (2.1). Then for* $f \in C_B[0,\infty)$ *we have*

$$|L^{\tau}_{n,q}(f;x) - f| \leq 2\omega\left(f;\delta_{n,\tau(x)}\right),$$

where $C_B[0,\infty)$ is the space of uniformly continuous bounded functions on $\mathbb{R}^+$, $\omega(f,\delta)$ is the modulus of continuity of the function $f \in C_B[0,\infty)$ defined in (4.1) and

$$\delta_{n,\tau(x)} = \sqrt{[1+2\mu]_q \frac{\tau(x)}{[n]_q}}. \tag{4.3}$$

Proof. We prove it by using (4.1), (4.2) and Cauchy-Schwarz inequality we can easily get

$$|L^{\tau}_{n,q}(f;x) - f| \leq \left\{1 + \frac{1}{\delta}\left(L^{\tau}_{n,q}((\tau(t)-\tau(x))^2;x\right)^{\frac{1}{2}}\right\}\omega(f;\delta)$$

if we choose $\delta = \delta_{n,\tau(x)}$ and by applying the result (2.2) of Lemma 2.2 complete the proof.

Remark 4.2. *For the operators* $D_{n,q}(.;.)$ *defined by (1.20) we may write that, for every* $f \in C_B[0,\infty)$, $x \geq 0$ *and* $n \in \mathbb{N}$

$$|D_{n,q}(f;x) - f(x)| \leq 2\omega\left(f;\lambda_{n,x}\right), \tag{4.4}$$

where by [16] we have

$$\lambda_{n,x} = \sqrt{D_{n,q}((t-x)^2;x)} \leq \sqrt{[1+2\mu]_q \frac{x}{[n]_q}}. \tag{4.5}$$

Now we claim that the error estimation in Theorem 4.1 is a generalization of (4.4) provided $f \in C_B[0,\infty)$ and $\tau(x)$ is a continuously differentiable function on $\mathbb{R}^+$. Indeed, for $\tau(x) = x$ and $x \in \mathbb{R}^+$, under these conditions, we notice that $|x-t| \leq |\tau(t)-\tau(x)|$, for every $x,t \in \mathbb{R}^+$. It is guarantees that

$$L^{\tau}_{n,q}((\tau(t)-\tau(x))^2;x) = D_{n,q}((t-x)^2;x). \tag{4.6}$$

Now we give the rate of convergence of the operators $L_{n,q}^{\tau}(f;x)$ defined in (2.1) in terms of the elements of the usual Lipschitz class $Lip_M(\nu)$.

Let $f \in C_B[0,\infty)$, $M > 0$ and $0 < \nu \leq 1$. In [1], the Lipschitz type maximal function space on $E \subset \mathbb{R}_+$ is defined as

$$\tilde{W}_{\alpha,E} = \{f : \sup(1+\tau(x))^{\alpha} \tilde{f}_{\alpha}(x) \leq M \frac{1}{(1+\tau(y))^{\alpha}} : and \ y \in E\} \tag{4.7}$$

where f is bounded and continuous function on $\mathbb{R}_+$, M is a positive constant, $0 < \alpha \leq 1$ and $\tau(x)$ continuously differentiable function on $\mathbb{R}^+$.

In [17], B.Lenze introduced a Lipschitz type maximal function f_α as follows:

$$f_{\alpha}(x,t) = \sum_{\substack{t>0 \\ t \neq x}} \frac{|f(t)-f(x)|}{|\tau(x)-\tau(t)|^{\alpha}}. \tag{4.8}$$

Theorem 4.3. *Let* $L_{n,q}^{\tau}(.;.)$ *be the operator defined in (2.1).Then for each* $f \in Lip_M(\nu)$, $(M > 0, \ \ 0 < \nu \leq 1)$ *satisfying (4.7) and (4.8) we have*

$$|L_{n,q}^{\tau}(f;x) - f| \leq M\left(\delta_{n,\tau(x)}\right)^{\frac{\nu}{2}}$$

where $\delta_{n,\tau(x)}$ is given in Theorem 4.1.

Proof. We prove it by using (4.7), (4.8) and Hölder inequality.

$$\begin{aligned} |L_{n,q}^{\tau}(f;x) - f| &\leq |L_{n,q}^{\tau}(f(t)-f;x)| \\ &\leq L_{n,q}^{\tau}\left(|f(t)-f|;x\right) \\ &\leq |ML_{n,q}^{\tau}\left(|\tau(t)-\tau(x)|^{\nu};x\right). \end{aligned}$$

Therefore

$$\begin{aligned} &|L_{n,q}^{\tau}(f;x) - f| \\ &\leq M \frac{[n]_q}{e_{\mu,q}([n]_q\tau(x))} \sum_{k=0}^{\infty} \frac{([n]_q\tau(x))^k}{\gamma_{\mu,q}(k)} \left| \frac{1-q^{2\mu\theta_k+k}}{1-q^n} - \tau(x)\right|^{\nu} \mathrm{d}t \\ &\leq M \frac{[n]_q}{e_{\mu,q}([n]_q\tau(x))} \sum_{k=0}^{\infty} \left(\frac{([n]_q\tau(x))^k}{\gamma_{\mu,q}(k)}\right)^{\frac{2-\nu}{2}} \\ &\times \left(\frac{([n]_q\tau(x))^k}{\gamma_{\mu,q}(k)}\right)^{\frac{\nu}{2}} \left| \frac{1-q^{2\mu\theta_k+k}}{1-q^n} - \tau(x)\right|^{\nu} \mathrm{d}t \end{aligned}$$

$$\leq M\left(\frac{n}{e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\frac{([n]_q\tau(x))^k}{\gamma_{\mu,q}(k)}\mathrm{d}t\right)^{\frac{2-\nu}{2}}$$

$$\times\left(\frac{[n]_q}{e_{\mu,q}([n]_q\tau(x))}\sum_{k=0}^{\infty}\frac{([n]_q\tau(x))^k}{\gamma_{\mu,q}(k)}\left|\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-\tau(x)\right|^2\mathrm{d}t\right)^{\frac{\nu}{2}}$$

$$= M\left(L^{\tau}_{n,q}(\tau(t)-\tau(x))^2;x\right)^{\frac{\nu}{2}}.$$

Which complete the proof.

Let $C_B[0,\infty)$ denote the space of all bounded and continuous functions on $\mathsf{R}^+=[0,\infty)$ and

$$C_B^2(\mathsf{R}^+)=\{g\in C_B(\mathsf{R}^+): g^{'},g^{''}\in C_B(\mathsf{R}^+)\}, \tag{4.9}$$

with the norm

$$\| g\|_{C_B^2(\mathsf{R}^+)}=\| g\|_{C_B(\mathsf{R}^+)}+\| g^{'}\|_{C_B(\mathsf{R}^+)}+\| g^{''}\|_{C_B(\mathsf{R}^+)}, \tag{4.10}$$

also

$$\| g\|_{C_B(\mathsf{R}^+)}=\sup_{x\in\mathsf{R}^+}|g|. \tag{4.11}$$

Theorem 4.4. *Let $L^{\tau}_{n,q}(.;.)$ be the operator defined in (2.1). Then for any $g\in C_B^2(\mathsf{R}^+)$ we have*

$$|L^{\tau}_{n,q}(g;x)-g|\leq\frac{\delta_{n,\tau(x)}}{2}\| g\|_{C_B^2(\mathsf{R}^+)},$$

where $\delta_{n,\tau(x)}$ is given in Theorem 4.1.

Proof. Let $g\in C_B^2(\mathsf{R}^+)$, then by using the generalized mean value theorem in the Taylor series expansion we have

$$g(\tau(t))=g+(\tau(t)-\tau(x)).g^{'}+\frac{(\tau(t)-\tau(x))^2}{2}.g^{''}(\psi),\quad \psi\in(\tau(t),\tau(x)).$$

By applying linearity property on $L^{\tau}_{n,q}$, we have

$$L^{\tau}_{n,q}(g,x)-g=g^{'}.L^{\tau}_{n,q}\left((\tau(t)-\tau(x));x\right)+\frac{g^{''}(\psi)}{2}.L^{\tau}_{n,q}\left((\tau(t)-\tau(x))^2;x\right),$$

which imply that

$$|L^{\tau}_{n,q}(g;x)-g|\leq\left([1+2\mu]_q\frac{\tau(x)}{[n]_q}\right)\frac{\| g^{''}\|_{C_B(\mathsf{R}^+)}}{2}.$$

From (4.10) we have $\| g^{''}\|_{C_B[0,\infty)}\leq\| g\|_{C_B^2[0,\infty)}$.

$$|L_{n,q}^{\tau}(g;x)-g|\leq\left([1+2\mu]_q\frac{\tau(x)}{[n]_q}\right)\frac{\|g\|_{C_B^2(\mathsf{R}^+)}}{2}.$$

This completes the proof from 2.2 of Lemma 2.2.
The Peetre's K-functional is defined by

$$K_2(f,\delta)=\inf_{C_B^2(\mathsf{R}^+)}\left\{\left(\|f-g\|_{C_B(\mathsf{R}^+)}+\delta\|g''\|_{C_B^2(\mathsf{R}^+)}\right):g\in\mathsf{W}^2\right\},\qquad(4.12)$$

where

$$\mathsf{W}^2=\left\{g\in C_B(\mathsf{R}^+):g',g''\in C_B(\mathsf{R}^+)\right\}.\qquad(4.13)$$

There exits a positive constant $C>0$ such that $K_2(f,\delta)\leq C\omega_2(f,\delta^{\frac{1}{2}}),\ \delta>0$, where the second order modulus of continuity is given by

$$\omega_2(f,\delta^{\frac{1}{2}})=\sup_{0<h<\delta^{\frac{1}{2}}}\sup_{x\in\mathsf{R}^+}|f(x+2h)-2f(x+h)+f(x)|.\qquad(4.14)$$

Theorem 4.5 *Let* $L_{n,q}^{\tau}(.;.)$ *be the operator defined in (2.1) and* $C_B[0,\infty)$ *be the space of all bounded and continuous functions on* R^+. *Then for any continuously differentiable function* $\tau(x)$ *on* R^+ *and* $f\in C_B(\mathsf{R}^+)$ *we have*

$$|L_{n,q}^{\tau}(f;x)-f|$$
$$\leq 2M\left\{\omega_2\left(f;\sqrt{\frac{\delta_{n,\tau(x)}}{4}}\right)+\min\left(1,\frac{\delta_{n,\tau(x)}}{4}\right)\|f\|_{C_B(\mathsf{R}^+)}\right\},$$

where M is a positive constant, $\delta_{n,\tau(x)}$ is given in Theorem 4.3 and $\omega_2(f;\delta)$ is the second order modulus of continuity of the function f defined in (4.14).

Proof. We prove this by using the Theorem 4.4

$$|L_{n,q}^{\tau}(f;x)-f|\leq|L_{n,q}^{\tau}(f-g;x)|+|L_{n,q}^{\tau}(g;x)-g|+|f-g|$$
$$\leq 2\|f-g\|_{C_B(\mathsf{R}^+)}+\frac{\delta_{n,\tau(x)}}{2}\|g\|_{C_B^2(\mathsf{R}^+)}$$

From (4.10) clearly we have $\|g\|_{C_B[0,\infty)}\leq\|g\|_{C_B^2[0,\infty)}$.

Therefore,

$$|L_{n,q}^{\tau}(f;x)-f|\leq 2\left(\|f-g\|_{C_B(\mathsf{R}^+)}+\frac{\delta_{n,\tau(x)}}{4}\|g\|_{C_B^2(\mathsf{R}^+)}\right),$$

where $\delta_{n,\tau(x)}$ is given in Theorem 4.1.

By taking infimum over all $g \in C_B^2(\mathbb{R}^+)$ and by using (4.12), we get

$$| L_{n,q}^{\tau}(f;x) - f | \leq 2K_2\left(f; \frac{\delta_{n,\tau(x)}}{4}. \right)$$

Now for an absolute constant $Q > 0$ in [7] we use the relation

$$K_2(f;\delta) \leq Q\{\omega_2(f;\sqrt{\delta}) + \min(1,\delta) \| f \|\}.$$

This complete the proof.

Conclusion.

Purpose of this paper is to introduce a modification of sequence of Dunkl generalization of exponential functions via q-calculus which is based on a continuously differentiable function τ on $[0,\infty)$ by $\tau(0) = 0$ and $\inf_{x\in\mathbb{R}^+} \tau'(x) \geq 1$. Here we have defined a Dunkl generalization of these modified operators. This type of modifications enables to obtain the degree of approximation by continuously differentiable function τ on the interval $\mathbb{R}^+$ rather than the classical Dunkl Szász operators via q-calculus [16]. We obtained some approximation results via well known Korovkin's type theorem. We have also calculated the rate of convergence of operators by means of modulus of continuity and Lipschitz type maximal functions.

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A Study On $(LCS)_n$-Manifolds Admitting η-Ricci Soliton.

K.R. Vidyavathi and C.S. Bagewadi

Department of Mathematics,
Kuvempu University, Shankaraghatta - 577 451, Shimoga, Karnataka, India.
e-mail:vidyarsajjan@gmail.com, prof_bagewadi@yahoo.co.in

Abstract We study the η-Ricci solitons on $(LCS)_n$-manifolds and show that they are Einstein when Riemannian curvature tensor satisfies pseudo-symmetric and semi-symmetric conditions $R\cdot\overline{P}=L_{\overline{P}}Q(g,\overline{P})$, $\overline{P}\cdot R=L_R Q(g,R)$ $R\cdot\overline{P}=0$ and $\overline{P}\cdot R=0$ where $\overline{P}$ is the pseudo projective curvature tensor.

Key Words: η-Ricci soliton, Lorentzian concircular structure, pseudo-projective curvature tensor.
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1. Introduction

The notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with an example was introduced by Shaikh [11], which generalize the notion of LP-Sasakian manifolds introduced by Matsumoto [7] and also by Mihai and Rosca [8]. Then Shaikh and Baishya [12], [13] investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. The $(LCS)_n$-manifolds are also studied by Atceken et al. [1], [2] and many authors.

In 1982 Hamilton [5] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. It has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman [9], [10] used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows.

$$\frac{\partial}{\partial t}g_{ij}(t)=-2R_{ij},$$

A Ricci soliton emerges as the limit of the solitons of the Ricci flow. A soliton to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g,V,λ) on a Riemannian manifold (M,g) is a generalization of Einstein metric such that [6]

$$L_V g+2S+2\lambda g=0,$$

where S is the Ricci tensor and L_V is the Lie derivative along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding as λ is negative, zero and positive respectively.

As a generalization of Ricci solitons, the notion of η-Ricci solitons was introduced by Cho and Kimura [3]. This notion has also been studied in [3] for Hopf hypersurfaces in complex space forms. An η-Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M. λ and μ are constants and g is a Riemannian metric satisfying the equation

$$L_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.1}$$

where S is the Ricci tensor associated to Riemannian metric g. In particular, if $\mu = 0$ then (g, V, λ) is Ricci soliton.

The interest in studying Ricci solitons has considerably increased and has been carried out in many contexts; on Kenmotsu manifolds, α-Sasakian manifolds, trans-Sasakian manifolds, Lorentzian α-Sasakian manifolds, $(LCS)_n$-manifolds, f-Kenmotsu manifolds respectively. Recently A. M. Blaga studied the η-Ricci soliton on Lorentzian para-Sasakian manifolds and on para-Kenmotsu manifolds. In this paper we use semi-symmetry, pseudo-symmetry conditions on pseudo-projective curvature tensor i.e., $R \cdot \overline{P} = L_{\overline{P}} Q(g, \overline{P})$, $R \cdot \overline{P} = 0$, $\overline{P} \cdot R = L_R Q(g, R)$ and $\overline{P} \cdot R = 0$ to study η-Ricci solitons of $(LCS)_n$ manifolds.

2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdarff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-,+,........,+)$ where T_pM denotes the tangent vector space of M at p and R is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v,v) < 0$ (resp., ≤ 0, $= 0$, > 0). The category to which a given vector falls is called its casual character.

Definition 1. In a Lorentzian manifold (M, g) a vector field P defined by

$$g(X, P) = A(X),$$

for any $X \in \chi(M)$ is said to be a concircular vector field if

$$(\nabla_X A)(Y) = \alpha\{g(X,Y) + \omega(X)A(Y)\},$$

where α is a non-zero scalar and ω is a closed 1-form.

Let M^n be a Lorentzian manifold admitting a unit timelike concircular vector field ξ and called it as the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1, \tag{2.1}$$

since ξ is a unit concircular vector field, it follows that there exits a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{2.2}$$

the equation for the following form holds.

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0, \tag{2.3}$$

that is

$$(\nabla_X \xi) = \alpha[X + \eta(X)\xi], \tag{2.4}$$

for all vector fields X, Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$(\nabla_X \alpha) = X\alpha = d\alpha(X) = \rho\eta(X), \tag{2.5}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.6}$$

then from (2.3) and (2.6) we have

$$\phi X = X + \eta(X)\xi, \tag{2.7}$$

for which it follows that ϕ is a symmetric $(1,1)$ tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M^n together with the unit timelike concircular vector field ξ, its associated 1-form η and $(1,1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$ manifold) [11]. Especially if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [7]. In a $(LCS)_n$-manifold, the following relations hold.

$$\eta(\xi) = -1,\ \phi\xi = 0,,\ g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.8}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.9}$$

$$(\nabla_X \phi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.10}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)\{n(Y)X - \eta(X)Y\}, \tag{2.11}$$

$$R(\xi,Y)Z = (\alpha^2 - \rho)\{g(Y,Z)\xi - \eta(Z)Y\}, \tag{2.12}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}. \tag{2.13}$$

3. η-Ricci soliton on $(LCS)_n$-manifolds.

Let $M(\phi,\xi,\eta,g)$ be an n-dimensional Lorentzian concicular structure manifold and let $(M,(g,\xi,\lambda,\mu))$ be a $(LCS)_n$ η-Ricci soliton. Then the relation (1.1) implies

$$(L_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0,$$

i.e.,

$$2S(X,Y) = -(L_\xi g)(X,Y) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y). \tag{3.1}$$

Here $L_\xi g$ denotes the Lie derivative of Riemannian metric g along a vector field ξ, by the definition of Lie derivative we have

$$(L_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X). \tag{3.2}$$

Using (2.4) in (3.2) we obtain

$$(L_\xi g)(X,Y) = 2\alpha\{g(X,Y) + \eta(X)\eta(Y)\}. \tag{3.3}$$

Using (3.1) and (3.3) we can write

$$S(X,Y) = (-\alpha - \lambda)g(X,Y) + (-\alpha - \mu)\eta(X)\eta(Y). \tag{3.4}$$

Thus we state the following theorem:

Theorem 1. *An $(LCS)_n$ η-Ricci soliton $(M,(g,\xi,\lambda,\mu))$ is an η-Einstein manifold.*

In particular if $\mu = 0$ in (3.4), then it reduces to

$$S(X,Y) = (-\alpha - \lambda)g(X,Y) - \alpha\eta(X)\eta(Y). \tag{3.5}$$

Thus we state the following:

Corollary 1. An $(LCS)_n$-Ricci soliton $(M,(g,\xi,\lambda))$ is an η-Einstein manifold.

4. η-Ricci soliton on pseudo-projective pseudo-symmetric $(LCS)_n$-manifolds.

An $(LCS)_n$ manifold M is said to be pseudo-symmetric if M satisfies the condition $R\cdot\overline{P}=L_{\overline{P}}Q(g,\overline{P})$. Where $L_{\overline{P}}$ is some smooth function on M and $\overline{P}$ is the pseudo-projective curvature tensor and it is given by

$$\overline{P}(X,Y)Z=aR(X,Y)Z+b[S(Y,Z)X-S(X,Z)Y]-\frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(Y,Z)X-g(X,Z)Y]. \quad (4.1)$$

Using (2.13), (3.5) in (4.1) we get

$$\eta(\overline{P}(X,Y)Z)=\left[a(\alpha^2-\rho)+b(-\alpha-\lambda)-\frac{r}{n}\left(\frac{a}{n-1}+b\right)\right][g(Y,Z)\eta(X)-g(X,Z)\eta(Y)], \quad (4.2)$$

$$\overline{P}(\xi,Y)Z=\left[a(\alpha^2-\rho)+b(-\alpha-\lambda)-\frac{r}{n}\left(\frac{a}{n-1}+b\right)\right][g(Y,Z)\xi-\eta(Z)Y]. \quad (4.3)$$

Let us consider

$$(R\cdot\overline{P})(U,V,Z;\xi,Y)=L_{\overline{P}}(Q(g,\overline{P})(U,V,Z;\xi,Y)) \quad (4.4)$$

L.H.S of (4.4) takes the form

$$(R\cdot\overline{P})(U,V,Z;\xi,Y)=R(\xi,Y)\overline{P}(U,V)Z-\overline{P}(R(\xi,Y)U,V)Z-\overline{P}(U,R(\xi,Y)V)Z-\overline{P}(U,V)R(\xi,Y)Z \quad (4.5)$$

Taking inner product of (4.5) with ξ and by virtue of (2.12) and (4.2) we can obtain

$$g((R\cdot\overline{P})(U,V,Z;\xi,Y),\xi)=-(\alpha^2-\rho)P(U,V,Z,Y)$$
$$+(\alpha^2-\rho)\left[a(\alpha^2-\rho)+b(-\alpha-\lambda)-\frac{r}{n}\left(\frac{a}{n-1}+b\right)\right][g(Y,U)g(V,Z)-g(Y,V)g(U,Z)] \quad (4.6)$$

R.H.S of (4.4) takes the form

$$Q(g,\overline{P})(U,V,Z;\xi,Y)=(\xi\wedge Y)\overline{P}(U,V)Z-\overline{P}((\xi\wedge Y)U,V)Z.$$
$$-\overline{P}(U,(\xi\wedge Y)V)Z-\overline{P}(U,V)R(\xi\wedge Y)Z \quad (4.7)$$

Taking inner product of (4.7) with ξ and by using the definition of endomorphism i.e., $(X\wedge Y)Z=g(Y,Z)X-g(X,Z)Y$ and (4.2) we can obtain

$$g(Q(g,\overline{P})(U,V,Z;\xi,Y),\xi)=-P(U,V,Z,Y)$$
$$+\left[a(\alpha^2-\rho)+b(-\alpha-\lambda)-\frac{r}{n}\left(\frac{a}{n-1}+b\right)\right][g(Y,U)g(V,Z)-g(Y,V)g(U,Z)]. \quad (4.8)$$

Using equations (4.6) and (4.8) in (4.4) we can get

Either $L_{\overline{P}}=(\alpha^2-\rho)$ or

$$-P(U,V,Z,Y)+\left[a(\alpha^2-\rho)+b(-\alpha-\lambda)-\frac{r}{n}\left(\frac{a}{n-1}+b\right)\right][g(Y,U)g(V,Z)-g(Y,V)g(U,Z)]=0. \quad (4.9)$$

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $U = Y = e_i$ in (4.9) and taking summation over i, $(1 \le i \le n)$, using equation (4.1) we get

$$S(V,Z) = \left[\frac{a(\alpha^2-\rho)+b(-\alpha-\lambda)}{a+b(n-1)}\right] g(V,Z). \tag{4.10}$$

Thus we can state the following:

Theorem 2. *A pseudo-projective pseudo-symmetric* $(LCS)_n$ η*-Ricci soliton* $(M,(g,\xi,\lambda,\mu))$ *is an Einstein manifold provided* $L_{\overline{P}} \neq (\alpha^2 - \rho)$.

Similarly we obtain the same result for pseudo-projective semi-symmetric $(LCS)_n$-manifold and we can state the following.

Corollary 2. An pseudo-projectively semi-symmetric $(LCS)_n$ η-Ricci soliton $(M,(g,\xi,\lambda,\mu))$ is an Einstein manifold.

In particular, if $\mu = 0$ in (3.4) comparing with (4.10) and contacting we get the value of λ as

$$\lambda = \frac{an(\alpha^2-\rho)}{n(b-1)} + \frac{\alpha(a+b(n-1))}{n(b-1)} - \alpha \tag{4.11}$$

Thus, we can state the following

Corollary 3. A Ricci soliton in pseudo-projective pseudo-symmetric manifolds is given by (4.11)

5. η-Ricci soliton on $(LCS)_n$-manifold admitting the pseudo-symmetric condition $\overline{P} \cdot R = L_R Q(g,R)$.

Let us consider

$$(\overline{P}(\xi,Y)\cdot R)(U,V)Z = L_R[((\xi \wedge Y)\cdot R)(U,V)Z] \tag{5.1}$$

which implies

$$\overline{P}(\xi,Y)R(U,V)Z - R(\overline{P}(\xi,Y)U,V)Z - R(U,\overline{P}(\xi,Y)V)Z - R(U,V)\overline{P}(\xi,Y)Z$$

$$= L_R[(\xi\wedge Y)R(U,V)Z - R((\xi\wedge Y)U,V)Z - R(U,(\xi\wedge Y)V)Z - R(U,V)(\xi\wedge Y)Z] \tag{5.2}$$

Taking inner product of (5.2) with ξ and using (4.3) and (2.13) we can get

Either $L_R = \left[a(\alpha^2-\rho)+b(-\alpha-\lambda) - \frac{r}{n}\left(\frac{a}{n-1}+b\right)\right]$, or

$$R(U,V,Z,Y) = (\alpha^2-\rho)[g(Y,U)g(V,Z) - g(Y,V)g(U,Z)]. \tag{5.3}$$

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $U = Y = e_i$ in (5.3) and taking summation over i, $(1 \le i \le n)$, we get

$$S(V,Z) = (\alpha^2-\rho)(n-1)g(V,Z). \tag{5.4}$$

Thus we can state the following:

Theorem 3. *An* $(LCS)_n$ η *-Ricci soliton* $(M,(g,\xi,\lambda,\mu))$ *admitting semi-symmetric condition* $\overline{P}\cdot R = L_R Q(g,R)$ *is an Einstein manifold provided* $L_R \neq \left[a(\alpha^2-\rho)+b(-\alpha-\lambda)-\frac{r}{n}\left(\frac{a}{n-1}+b\right)\right]$

Corollary 4. *An* $(LCS)_n$ η *-Ricci soliton* $(M,(g,\xi,\lambda,\mu))$ *admitting semi-symmetric condition* $\overline{P}\cdot R = 0$ *is an Einstein manifold.*

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Contact CR-submanifolds of generalized Sasakian-space-forms

Shyamal Kumar Hui

Department of Mathematics, Sidho Kanho Birsha University, Purulia–723104, West Bengal, India
Present address
Department of Mathematics, The University of Burdwan, Burdwan–713104, West Bengal, India
E-mail: shyamal_hui@yahoo.co.in

The present paper deals with study of contact CR-submanifolds of generalized Sasakian-space-forms. The contact CR-products of generalized Sasakian-space-forms is also studied and found a necessary and sufficient condition for a contact CR-submanifold of a generalized Sasakian-space-form to be a contact CR-product.

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1 Introduction

It is well known that in differential geometry the curvature of a Riemannian manifold plays a basic role and the sectional curvatures of a manifold determine the curvature tensor R completely. A Riemannian manifold with constant sectional curvature c is called a real-space-form and its curvature tensor R satisfies the condition

$$\overline{R}(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}. \tag{1.1}$$

Models for these spaces are the Euclidean spaces ($c = 0$), the spheres ($c > 0$) and the hyperbolic spaces $(c < 0)$.

In contact metric geometry, a Sasakian manifold with constant ϕ-sectional curvature is called Sasakian-space-form and the curvature tensor of such a manifold is given by

$$\begin{aligned}\overline{R}(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \frac{c-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}.\end{aligned} \tag{1.2}$$

These spaces can also be modeled depending on $c > -3$, $c = -3$ or $c < -3$.

As a generalization of Sasakian-space-form, in [1] Alegre, Blair and Carriazo introduced and studied the notion of generalized Sasakian-space-form with the existence of such notions by several interesting examples. An almost contact metric manifold $\overline{M}(\phi,\xi,\eta,g)$ is called generalized

Sasakian-space-form if there exist three functions f_1, f_2, f_3 on $\overline{M}$ such that [1]

$$\begin{aligned}\overline{R}(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\end{aligned} \tag{1.3}$$

for all vector fields X, Y, Z on $\overline{M}$, where $\overline{R}$ is the curvature tensor of $\overline{M}$ and such a manifold of dimension $(2n+1)$, $n>1$ (the condition $n>1$ is assumed throughout the paper), is denoted by $\overline{M}^{2n+1}(f_1, f_2, f_3)$. The generalized Sasakian-space-forms have been studied by several authors such as Alegre and Carriazo ([2], [3], [4]), Belkhelfa, Deszcz and Verstraelen [8], Carriazo [10], C i rnu [12], Ghefari, Solamy and Shahid [13], Gherib, Gorine and Belkhelfa [14], Hui and Sarkar [16], Kim [18], Narain, Yadav and Dwivedi [20], Olteanu ([21], [22]), Shukla and Chaubey [23], Yadav, Suthar and Srivastava [25] and many others.

As a generalization of invariant and anti-invariant submanifolds, Bejancu [7] introduced and studied the notion of contact CR-submanifolds. The geometry of contact CR-submanifolds are rich and interesting subject. Several authors studied contact CR-submanifolds of different classes of almost contact metric manifolds such as [5], [6], [17], [19] and many others.

Motivated by the above studies the present paper deals with the study of contact CR-submanifolds of generalized Sasakian-space-forms. The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of contact CR-submanifolds of generalized Sasakian-space-forms. We obtain many integrability conditions of the distributions of contact CR-submanifolds of generalized Sasakian-space-forms. We also studied the contact CR-products in a generalized Sasakian-space-form and obtained a necessary and sufficient condition for a contact CR-submanifold of a generalized Sasakian-space-form to be a contact CR-product.

2. Preliminaries

In an almost contact manifold, we have [9]

$$\phi^2(X) = -X + \eta(X)\xi, \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, g(X,\xi) = \eta(X), \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.4}$$

$$(\overline{\nabla}_X \eta)(Y) = g(\overline{\nabla}_X \xi, Y), \tag{2.5}$$

where $\overline{\nabla}$ is a connection on $\overline{M}$.

In a generalized Sasakian-space-form $\overline{M}^{2n+1}(f_1, f_2, f_3)$, we have from (1.3) that

$$(\overline{\nabla}_X \phi)(Y) = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{2.6}$$

$$\overline{\nabla}_X \xi = -(f_1 - f_3)\phi X \tag{2.7}$$

$$\overline{S}(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y), \tag{2.8}$$

$$\overline{r} = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.9}$$

$$\overline{R}(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} \tag{2.10}$$

$$\overline{R}(\xi,X)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\}, \tag{2.11}$$

$$\eta(\overline{R}(X,Y)Z) = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{2.12}$$

$$\overline{S}(X,\xi) = 2n(f_1 - f_3)\eta(X). \tag{2.13}$$

Let M be a submanifold of a generalied Sasakian-space-form $\overline{M}^{2n+1}(f_1, f_2, f_3)$. Also let ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M respectively. Then the Gauss Waingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.14}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.15}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of M into $\overline{M}^{2n+1}(f_1, f_2, f_3)$. The second fundamental form h and the shape operator A_V are related by [26]

$$g(h(X,Y),V) = g(A_V X, Y). \tag{2.16}$$

for any submanifold M of a Riemannian metric on $\overline{M}^{2n+1}(f_1, f_2, f_3)$, the equation of Gauss is given by

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \\ &\quad + (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z)\end{aligned} \tag{2.17}$$

for any $X, Y, Z \in \Gamma(TM)$ where $\overline{R}$ and R denotes the Riemannian curvature tensors of

$\overline{M}$ and M respectively. The covariant derivative $\overline{\nabla}h$ of h is defined by

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y), \tag{2.18}$$

and the covariant derivative $\overline{\nabla}_X A_V$ is defined by

$$(\overline{\nabla}_X A_V)Y = \nabla_X (A_V Y) - A_{\nabla^{\perp}_X V} Y - A_V \nabla_X Y \tag{2.19}$$

for any $X, Y, Z \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. The normal part $(\overline{R}(X,Y)Z)^{\perp}$ of $\overline{R}(X,Y)Z$ from (2.17) is given by

$$(\overline{R}(X,Y)Z)^{\perp} = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z), \tag{2.20}$$

which is known as Codazzi equation. If in particular $\overline{R}(X,Y)Z)^{\perp} = 0$ then M is said to be curvature invariant submanifold of $\overline{M}$.
The Ricci equation is given by

$$g(\overline{R}(X,Y)V,U) = g(R^{\perp}(X,Y)V,U) + g([A_U, A_V]X, Y) \tag{2.21}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$, where $R^{\perp}$ denotes the Riemannian curvature tensor of the normal vector bundle $T^{\perp}M$ and if $R^{\perp} = 0$ then the normal connection of M is called flat [17].
Using (1.3) in (2.21) we get

$$\begin{aligned} g(R^{\perp}(X,Y)V,U) = {} & f_2[g(X,\phi V)g(\phi Y,U) - g(Y,\phi V)g(\phi X,U) \\ & + 2g(X,\phi Y)g(\phi V,U)] \\ & + g([A_V, A_U]X, Y) \end{aligned} \tag{2.22}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$.

3. Contact CR-submanifolds in generalized Sasakian-space-forms

Let M be an isometrically immersed submanifold of a generalized Sasakian-space-form $\overline{M}^{2n+1}(f_1, f_2, f_3)$. Then for any $X \in \Gamma(TM)$ we have

$$\phi X = EX + FX, \tag{3.1}$$

where EX and FX are the tangential and normal components of ϕX respectively. Also for any $V \in \Gamma(T^{\perp}M)$ we can write

$$\phi V = BV + CV, \tag{3.2}$$

where BV and CV are the tangential and normal components of ϕV respectively. Also B is an endomorphism of the normal bundle $T^{\perp}$ of TM and C is an endomorphism of the subbundle of the normal bundle $T^{\perp}M$.
The covariant derivatives of the tensor fields of E and F are defined by

$$(\nabla_X E)Y = \nabla_X EY - E(\nabla_X Y), \tag{3.3}$$

and

$$(\nabla_X F)Y = \nabla_X^{\perp} FY - F(\nabla_X Y) \tag{3.4}$$

for all $X, Y \in \Gamma(TM)$. The canonical structure E and F on a submanifold M are said to be parallel if $\nabla E = 0$ and $\nabla F = 0$ respectively. Also the covariant derivatives of B and C are defined by

$$(\nabla_X B)V = \nabla_X BV - B(\nabla_X^{\perp} V) \tag{3.5}$$

and

$$(\nabla_X C)V = \nabla_X^{\perp} CV - C(\nabla_X^{\perp} V). \tag{3.6}$$

Also for any $X, Y \in \Gamma(TM)$, we have $g(EX, Y) = -g(X, EY)$ and for any $U, V \in \Gamma(T^{\perp}M)$, we have $g(U, CV) = -g(CU, V)$. This shows that E and C are skew symmetric tensor fields.

Moreover for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, the relation between F and B is given by

$$g(FX, V) = g(X, BV) \tag{3.7}$$

Definition 3.1. *[6] Let M be an isometrically immersed submanifold of a generalized Sasakian-space-form $\overline{M}^{2n+1}(f_1, f_2, f_3)$. Then M is called a contact CR-submanifold of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ if there is a differentiable distribution $D : p \to D_p \subseteq T_p(M)$ on M satisfying the following conditions*

(i) $\xi \in D$

(ii) D is invariant with respect to ϕ i.e. $\phi(D_p) \subset D_p$ for each $p \in M$, and

(iii) The orthogonal complementary distribution $D^{\perp} : p \to D_p \subseteq T_p(M)$ satisfies $\phi(D_p^{\perp}) \subseteq T_P^{\perp} M$ for each $p \in M$

Now from (2.1)

$$\phi\xi = E\xi + F\xi = 0,$$

which is euqivalent to

$$E\xi = F\xi = 0 \tag{3.8}$$

Apply ϕ to (3.1) and using (2.1) and (3.2) we get

$$E^2 + BF = -I + \eta \otimes \xi \text{ and } EF + CF = 0. \tag{3.9}$$

Similarly apply ϕ to (3.2) and using (2.1) and (3.1) we get

$$C^2 + FB = -I \; and \; EB + BC = 0. \tag{3.10}$$

In view of (1.3) it follows from (2.17) that the curvature tensor R of an immersed submanifold M of a generalized Sasakian-space-form $\overline{M}^{2n+1}(f_1, f_2, f_3)$ is

$$R(X,Y)Z = f_1[g(Y,Z)X - g(X,Z)Y] \tag{3.11}$$

$$+ f_2[g(X,\phi Z)EY - g(Y,\phi Z)EX + 2g(X,\phi Y)EZ]$$
$$+ f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi$$
$$- g(Y,Z)\eta(X)\xi] + A_{h(Y,Z)}X - A_{h(X,Z)}Y.$$

From (1.3) and (2.20) we have

$$(\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z) = f_2[g(X,\phi Z)FY - g(Y,\phi Z)FX + 2g(X,\phi Y)FZ]. \tag{3.12}$$

Theorem 3.1. *There is no any curvature invariant proper contact CR-submanifold of a generalized Sasakian-space-form* $\overline{M}^{2n+1}(f_1, f_2, f_3)$ *such that* $f_2 \neq 0$.

Proof. Let M be a curvature invariant contact CR-submanifold of a generalized Sasakian-space-form $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with $f_2 \neq 0$. Then from (3.12) we get

$$g(X,EZ)FY - g(Y,EZ)FX + 2g(X,EY)FZ = 0, \tag{3.13}$$

for any $X,Y,Z \in \Gamma(TM)$. Putting $Z = X$ in (3.13) we get

$$3g(EY,X)FX = 0,$$

which implies that either $F = 0$ or $E = 0$, that is either M is invariant or anti-invariant submanifold.

Theorem 3.2. *Let* M *be a contact CR-submanifold of a generalized Sasakian-space-form* $\overline{M}^{2n+1}(f_1, f_2, f_3)$ *with flat normal connection and let* $f_2 \neq 0$. *If* $EA_V = A_V E$ *for any vector* $V \in \Gamma(T^{\perp}M)$ *then either* M *is an anti-invariant submanifold or generic submanifold of* $\overline{M}^{2n+1}(f_1, f_2, f_3)$.

Proof. Let the normal connection of M be flat then from (2.22) we obtain

$$g([A_U, A_V]X,Y) = f_2[g(X,\phi V)g(\phi Y,U) - g(Y,\phi V)g(\phi X,U) + 2g(X,\phi Y)g(\phi V,U)] \tag{3.14}$$

for any $X,Y \in \Gamma(TM)$ and $U,V \in \Gamma(T^{\perp}M)$. Taking $X = EY$ and $V = CU$ in (3.14) we get

$$g(A_U, A_{CU}EY - A_{CU}A_U EY,Y) = 2f_2[g(E^2Y,Y)g(CU,CU)]. \tag{3.15}$$

If $EA_U = A_U E$ then we have $f_2\mathrm{tr}(E^2)g(CU,CU) = 0$ i.e., $\mathrm{tr}(E^2)g(CU,CU) = 0$ as $f_2 \neq 0$, either which implies that $E = 0$ that means M is anti-invariant submanifold or $CV = 0$ that means M is generic submanifold.

Theorem 3.3 *Let* M *be a contact CR-submanifold of a generalized Sasakian-space-form* $\overline{M}^{2n+1}(f_1, f_2, f_3)$. *Then the invariant distribution* D *is integrable if and only if the second fundamental form of* M *satisfies* $h(X,\phi Y) = h(\phi X,Y)$ *for any* $X,Y \in \Gamma(D)$.

Proof. For any vector fields X, Y in D we have from (2.6) and (2.14) that

$$\begin{aligned}\phi[X,Y] &= \phi(\nabla_X Y - \nabla_Y X)\\ &= \phi(\overline{\nabla}_X Y - \overline{\nabla}_Y X)\\ &= \overline{\nabla}_X \phi Y - (\overline{\nabla}_X \phi)Y - \overline{\nabla}_Y \phi X + (\overline{\nabla}_Y \phi)X\\ &= \nabla_X \phi Y - \nabla_Y \phi X + h(X,\phi Y) - h(Y,\phi X) + (f_1 - f_3)[\eta(Y)X - \eta(X)Y].\end{aligned} \tag{3.16}$$

Comparing the normal components of (3.16) we get

$$F[X,Y] = h(X,\phi Y) - h(Y,\phi X). \tag{3.17}$$

Thus D is integrable if and only if $h(X,\phi Y) = h(Y,\phi X)$ for any $X, Y \in \Gamma(D)$.

Definition 3.2. *If the invariant distribution D and anti-invariant distribution $D^{\perp}$ are totally geodesic in M then M is called contact CR-product.*

Now we characterize contact CR-products in generalized Sasakian-space-form $\overline{M}^{2n+1}(f_1, f_2, f_3)$.

Theorem 3.4. *Let M be a contact CR-submanifold of a generalized Sasakian-space-form $\overline{M}^{2n+1}(f_1, f_2, f_3)$. Then M is a contact CR-product if and only if the shape operator A of M satisfies the condition*

$$A_{\phi W}\phi X + (f_1 - f_3)\eta(X)\phi W = 0 \tag{3.18}$$

for all $X \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$.

Proof. Let us take M be a contact CR-submanifold of a generalized Sasakian-space-form $\overline{M}^{2n+1}(f_1, f_2, f_3)$. Then for all $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$ we have from (2.3), (2.6), (2.14) and (2.16) we get

$$\begin{aligned}g(A_{\phi W}\phi X, Y) &= g(h(\phi X, Y), \phi W)\\ &= g(\overline{\nabla}_Y \phi X, \phi W)\\ &= g((\overline{\nabla}_Y \phi)X + \phi\overline{\nabla}_Y X, \phi W)\\ &= g((f_1 - f_3)\{g(X,Y)\eta - \eta(X)Y\}, \phi W) + g(\nabla_Y X, W)\\ &= -(f_1 - f_3)\eta(X)g(Y,\phi W) + g(\nabla_Y X, W)\end{aligned}$$

i. e.,

$$g(A_{\phi W}\phi X + (f_1 - f_3)\eta(X)\phi W, Y) = g(\nabla_Y X, W) \tag{3.19}$$

and

$$\begin{aligned}g(A_{\phi W}\phi X, Z) = g(h(\phi X, Z), \phi W) &= g(\overline{\nabla}_Z \phi X, \phi W)\\ &= g((\overline{\nabla}_Z \phi)X + \phi\overline{\nabla}_Z X \phi W)\\ &= g((f_1 - f_3)\{g(X,Z)\xi - \eta(X)Z\}, \phi W) + g(\overline{\nabla}_Z X, W)\\ &= -(f_1 - f_3)\eta(X)g(Z,\phi W) - g(\nabla_Z W, X),\end{aligned}$$

i. e.,

$$g(A_{\phi W}\phi X + (f_1 - f_3)\eta(X)\phi W, Z) = -g(\nabla_Z W, X). \qquad (3.20)$$

Thus from (3.19) and (3.20) we get $\nabla_Y X \in \Gamma(D)$ and $\nabla_Z W \in \Gamma(D^{\perp})$ if and only if the relation (3.18) holds.

Theorem 3.5. *Let M be a contact CR-submanifold of a generalized Sasakian-space-form $\overline{M}^{2n+1}(f_1, f_2, f_3)$. Then the anti-invariant distribution $D^{\perp}$ is always integrable.*

Proof. For any vector fields Z, W belongs to $D^{\perp}$ we have from (2.16) that

$$f(\nabla_Z W) = -Bh(Z,W) - A_{\phi W}Z,$$

i. e.,

$$f[Z,W] = A_{\phi Z}W - A_{\phi W}Z. \qquad (3.21)$$

On the other hand, we obtain

$$\begin{aligned} g(A_{\phi W}Z,U) &= g(h(U,Z),\phi W) \\ &= -g(\phi(\overline{\nabla}_U Z),W) \\ &= -g(\overline{\nabla}_U \phi Z - (\overline{\nabla}_U \phi)Z,W) \\ &= g(A_{\phi Z}W,U) + (f_1 - f_3)[g(U,Z)\eta(W) - \eta(Z)g(U,W)] \\ &= g(A_{\phi Z}W,U) \ \textit{for any}\ U \in \Gamma(TM). \end{aligned}$$

This implies that

$$A_{\phi Z}W = A_{\phi W}Z, \ \textit{for any vector fields}\ Z,W \in \Gamma(D^{\perp}). \qquad (3.22)$$

In view of (3.22) we have from (3.21) that $f[Z,W] = 0$, that is $[Z,W] \in \Gamma(D^{\perp})$ which completes the proof.

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