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Fourier Transform in communication system and Applications

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Abstract: The Fourier Transform is a tool that breaks a waveform (a function or signal) into an alternate representation, characterized by sine and cosines. The Fourier Transform shows that any waveform can be re-written as the sum of sinusoidal functions. Signal transmission is done through modulation. At the receiving end the transmitted signal is demodulated the information. All these techniques are based on sinusoidal function. While modulating the information signal a high frequency sinusoidal carrier signal is used to transmit the information signal through a medium. It then received and demodulated using Fourier Transform analysis. In this article, we discuss amplitude modulated signals with an example.

Keyword: Fourier Transform, Modulation, Demodulation.

1. Introduction

The present era of communication technology has provided some major catalysts in developing the modern human society. Communication includes automatic transmission of data over wires and radio circuits through signals. Signal is basically a means of transmitting information in accordance with certain prearranged system or code. It includes audio, video, image etc.

As we shall also come to argue later, what we shall call the time and frequency domains immediately relate to the ways in which the human ear and eye interpret stimuli. The ear, for example, responds to minute variations in atmospheric pressure. These cause the ear drum to vibrate and, the various nerves in the inner ear then convert these vibrations into what the brain interprets as sounds. In the eye, by contrast, electromagnetic waves fall on the rods and cones in the back of the eyeball, and are converted into what the brain interprets as colours. But there are fundamental differences in the way in which these interpretations occur. Specifically, consider one of the great American pastimes - watching television. The speaker in the television vibrates, producing minute compressions and rarefactions (increases and decreases in air pressure), which propagate across the room to the viewer’s ear. These variations impact on the ear drum as a single continuously varying pressure. However, by the time the result is interpreted by the brain, it has been separated into different actors’ voices, the background sounds, etc. That is, the nature of the human ear is to take a single complex signal (the sound pressure wave), decompose it into simpler components, and recognize the simultaneous existence of those different components. Fourier transform is a mathematical tool that breaks a function, a signal or a waveform into an another representation which is characterized by sin and cosines. In the theory of communication a signal is generally a voltage and Fourier transform is essential mathematical tool which provides us an inside view of signal and its different domain. Mathematically speaking, The Fourier transform is a linear operator that maps a functional space to another functions space and decomposes a function into another function of its frequency components. This is the essence of what we shall come to view, in terms of Fourier analysis, as frequency domain analysis of a signal (see [2, 3, 4, 6, 7, 9]).

Modulation: In order to carry the audio signal message to large distance, it is superimposed on high frequency carrier wave. The process is called modulation.

Two types of modulation will be reviewed in this module. Amplitude modulation consist encoding information onto a carrier signal by varying the amplitude of the carrier. Frequency modulation consists of encoding information onto a carrier signal by varying the frequency of the carrier. Once a signal has been modulated, information is retrieved through a demodulation process.

Amplitude modulation: In amplitude modulation, the amplitude of modulated wave varies in accordance with amplitude of information wave. When amplitude of information increases, the amplitude of modulated wave increases and vice versa. In this case the amplitude of modulated wave is not constant, as show below:

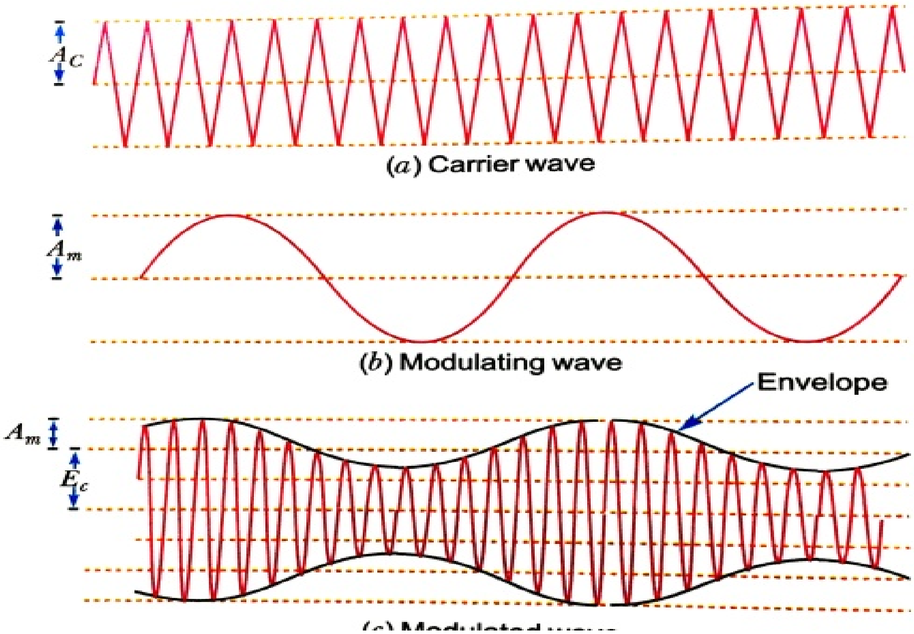


Fig -1

Modulation index: The modulation index of an amplitude modulated wave is defined as the ratio of the amplitude modulating signal to the amplitude of carrier wave.

i.e., $m_a = \frac{A_m}{A_c}$

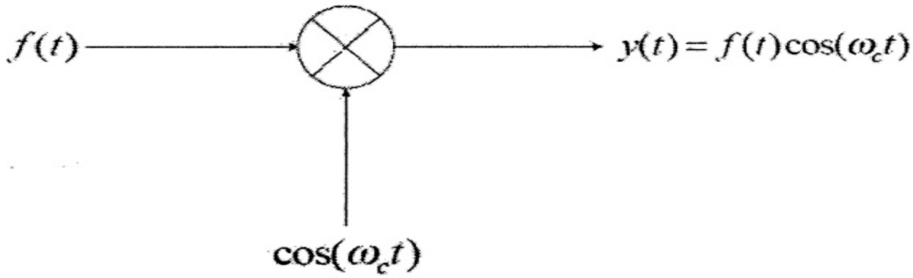
For modulated wave,

$$m_a = \frac{A_{max} - A_{min}}{A_{max} + A_{min}}$$

Double side band:

A general sinusoidal signal can be expressed as f(t) = A(t) cosθ(t), where A is amplitude. It is convenient to write time varying angle $\theta(t)$ as $\omega_c(t) + \emptyset(t)$

Therefore, the sinusoidal signal may be expressed as $f(t) = A(t)cos[\omega_c(t) + \emptyset(t)]$



Amplitude modulation

Fig - 2

Where A(t) is called the envelope of signal f(t), and ω_c is called the carrier frequency. A mathematical representation of an amplitude modulated signal is obtained by setting = 0 in the expression for the general sinusoidal signal, and letting the envelope A(t) be proportional to a modulating signal f(t). What result is new signal given by

$$y(t) = f(t)\cos(\omega_c t).$$

The spectrum of the modulated signal y(t) can be found by using the modulation property of Fourier transforms. The Fourier transform pair was defined as,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{j\omega t}d\omega,$$

$$F(t) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt.$$

The Fourier transform of a signal $f(t)e^{-j\omega t}$ is then

$$F[f(t)e^{j\omega_0 t}] = \int_{-\infty}^{+\infty} f(t)e^{j\omega_0 t}.e^{-j\omega t}dt,$$

$$F[f(t)e^{j\omega_0 t}] = \int_{-\infty}^{+\infty} f(t)e^{-j(\omega-\omega_0)t}dt.$$

Thus, the Fourier transform of $f(t)e^{j\omega_0 t}$ may be expressed

$$F[f(t)e^{j\omega_0 t}] = F(\omega-\omega_0).$$

The amplitude modulated signal y(t) may be written in terms of complex exponentials

$$y(t) = f(\mathrm{t})\cos(\omega_c \mathrm{t}) = \frac{1}{2}f(t)[e^{j\omega_0 t} + e^{-j\omega_c t}].$$

When y(t) is expressed in this form, and form the example above, it can be seen that the Fourier transform of y(t) is given by

$$F[f(\mathrm{t})\cos(\omega_c \mathrm{t})] = \frac{1}{2}[F(\omega+\omega_c) + F(\omega-\omega_c)].$$

Thus, the spectrum of f(t) is translated by $\pm\omega_c$.

It is seen that the modulating process causes frequency associated with modulating signal to disappear. Instead, a new frequency spectrum appears, consisting of two sidebands, known as the upper sideband (USB), and the lower side band (LSB). The spectrum of the original carrier, but is still cantered about carrier frequency ω_c. Thus this type of modulation is referred to as double-side band, suppressed-carrier amplitude modulation as shown in fig 2.

If the modulating signal contains a single frequency, ω_m, then $\omega_{USB} = \omega_c + \omega_m$ and $\omega_{LSB} = \omega_c - \omega_m$ (In fig.3). If modulating signal f(t) has a bandwidth of ω_{bw}. The upper sideband of the spectrum of the modulated signal Y(ω) will extended from ω_c to $\omega_c + \omega_{bw}$. Likewise, the lower sideband will extend from $\omega_c - \omega_{bw}$ to ωc. Both the negative and positive frequency components of the modulating signal f(t) appear as positive frequencies in the spectrum of the modulated signal y(t). It is also seen that the bandwidth of f(t) is double in the spectrum of the modulated signal when this type of modulation is employed. For detail we can see([1,5,8,10,11,12]).

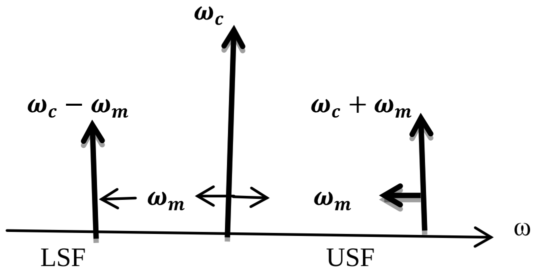


Fig – 3 A M signal frequency Spectrum

AM DEMODULATION

An AM signal is demodulated by first mixing the modulated signal y(t) with another sinusoid of the same carrier frequency,

$$y(t)\cos(\omega_c t) = f(t)\cos 2(\omega_c t)$$
$$= \frac{1}{2} f(t)(1 + \cos(2\omega_c t)).$$

The Fourier transform of the signal is

$$F(y(t)\cos(\omega_c t) = F(\tfrac{1}{2} f(t)(1 + \cos(2\omega_c t)),$$
$$F(y(t)\cos(\omega_c t) = \frac{1}{2}\left\{F(\omega) + \frac{1}{2}[(F(\omega + 2\omega_c) + F(\omega + 2\omega_c)]\right\}.$$

By using a low-pass filter, the frequency components cantered at ± ω_c can be removed to leave only the $\frac{1}{2}F(\omega)$ term. It is obvious that in order to properly recover the original signal it is necessary that $\omega_c > \omega_b$w.

The amplitude modulated in terms of complex exponentials carrier function ω_c.

The result is new signal given by

$$v(t) = e^{j\omega_c t}$$
$$\breve{y}(t) = \breve{x}(t)\, v(t).$$
$$V(j\omega) = 2\pi\delta(\omega - \omega_c).$$

$$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega_0)V(j(\omega - \omega_0)d\omega_0$$

$$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega_0)\delta(j(\omega - \omega_0 - \omega_c)d\omega_0 = X(j(\omega - \omega_c))$$
$$\breve{y}(t) = \breve{x}(t)\, cos(\omega_c t) + j\breve{x}(t)\, sin(\omega_c t).$$

When $u(t) = \breve{x}(t)\, cos(\omega_c t)$ and $w(t) = u(t)\, cos(\omega_c t)$, then

$$w(t) = \breve{x}(t)\; cos^2(\omega_c t) = \breve{x}(t)\,\frac{(1+cos(2\omega_c t))}{2},$$
$$V(j\omega) = \pi\delta(\omega + \omega_c) + \pi\delta(\omega - \omega_c).$$
$$W(j\omega) = \frac{1}{2}Y(j\omega + j\omega_c) + \frac{1}{2}Y(j\omega - j\omega_c)$$

$$W(j\omega) = \frac{1}{2}X(j\omega + j\omega_c) + \frac{1}{2}X(j\omega - j\omega_c).$$

If we take $v(t) = cos(\omega_c t)$, $\check{y}(t) = \check{x}(t)\, v(t) = \check{x}(t)\, cos(\omega_c t)$,

and $V(j\omega) = \pi\delta(\omega + \omega_c) + \pi\delta(\omega - \omega_c)$,

then we have

$$Y(j\omega) = \frac{1}{2}X(j\omega + j\omega_c) + \frac{1}{2}X(j\omega - j\omega_c).$$

Now, we state our main result.

Theorem: *If F be amplitude and ω be angular frequency of the periodic wave, then amplitude modulated in terms of sine waves of progressively increasing frequencies for the basic functions $y(t) = sin(n\omega_c t)$, where $n = 1,2,3, \dots$, is given by*

$$f[F(\omega)e^{j\omega t_0}] + f[F(\omega)e^{-j\omega t_0}] = f(t + t_0) + f(t - t_0).$$

Proof: First, we assume that a new signal given by

$$\check{y}(t) = \check{f}(t)\sin(n\omega_c t).$$

Then, the spectrum of the modulated signal y(t) can be found by using the modulation property of Fourier transforms.

The Fourier transform of a signal $\check{f}(t)e^{-j\omega t}$ is then given by

$$\check{F}\left[\check{f}(t)e^{jn\omega_0 t}\right] = \int_{-\infty}^{+\infty} \check{f}(t)e^{jn\omega_0 t}. e^{-j\omega t}dt,$$

$$\check{F}\left[\check{f}(t)e^{jn\omega_0 t}\right] = \int_{-\infty}^{+\infty} \check{f}(t)e^{-j(\omega - n\omega_0)t}dt$$

Thus, the Fourier transforms of $\check{f}(t)e^{jn\omega_0 t}$ and $\check{f}(t)e^{-jn\omega_0 t}$ can be expressed as

$$\check{F}\left[\check{f}(t)e^{jn\omega_0 t}\right] = \check{F}(\omega - \mathrm{n}\omega_0),$$

$$\check{F}\left[\check{f}(t)e^{-jn\omega_0 t}\right] = \check{F}(\omega + \mathrm{n}\omega_0).$$

The amplitude modulated signal $\check{y}$ (t) is written in terms of complex exponentials as

$$\check{y}(t) = \check{f}(t)\sin(n\omega_c t) = \frac{1}{2\mathrm{j}}\check{f}(t)[e^{jn\omega_0 t} - e^{-jn\omega_c t}].$$

If $\check{y}$ (t) is expressed in the above form, then it can be seen that the Fourier transform of y(t) is given by

$$\check{F}\left[\check{f}(t)\sin(n\omega_c t)\right] = \frac{1}{2\mathrm{j}}\left[\check{F}(\omega + n\omega_c) - \check{F}(\omega - n\omega_c)\right].$$

Now, for $n = 1$, we have

$$\check{F}\left[\check{f}(t)\sin(\omega_c t)\right] = \frac{1}{2\mathrm{j}}\left[\check{F}(\omega + \omega_c) - \check{F}(\omega - \omega_c)\right]$$

and $$F[f(t)\cos(\omega_c t)] = \frac{1}{2}[F(\omega + \omega_c) + F(\omega - \omega_c)].$$

Also it is easy to see

$$f\left[F(\omega)e^{-j\omega t_0}\right] = f(t - t_0),$$

$$f\left[F(\omega)e^{j\omega t_0}\right] = f(t + t_0),$$

and $$f\left[F(\omega)e^{j\omega t_0}\right] + f\left[F(\omega)e^{-j\omega t_0}\right] = f(t + t_0) + f(t - t_0),$$

which proves the result.

Now, we illustrate the result by the following example.

Example:

$$h(t) = v(t)cos5t, \quad v(t) = 6cost - 2sin3t.$$

In above case the analytic signal is defined as follows:

$$h_+(t) = h(t) + j\check{h}(t) = \cos(\omega_c t) + jsin(\omega_c t) = e^{j\omega_c t}$$
$$h_-(t) = h(t) - j\check{h}(t) = \cos(\omega_c t) - jsin(\omega_c t) = e^{-j\omega_c t}.$$

Therefore, $h(t) = (6cost - 2sin3t)cos5t.$

$$h(t) = 3cos6t + 3cos4t - 8sin8t + sin2t.$$
$$\check{h}(t) = 3sin6t + 3sin4t + 8cos8t - cos2t$$
$$h_+(t) = 3(cos6t + jsin6t) + 3(cos4t + jsin4t)$$
$$+8(-sin8t + jcos8t) + sin2t - jcos2t$$

Hence,

$$h_+(t) = (6cost - 2sin3t)e^{j5t}$$
$$h_-(t) = (6cost - 2sin3t)e^{-j5t}.$$

2. CONCLUSIONS

Finally, we have the following conclusions:

1. The Fourier Transform can be used to analyse and demonstrate different modulation signal. Such as Amplitude Modulation and Frequency modulation. Modulation techniques are made easier with the help of Fourier Transform which transforms time domain into frequency domain. In modulation technique the modulation theorem is of great importance in radio and television where harmonic carrier wave is modulated by an envelope.
2. Fourier Transform plays a vital and important role. In Digital Signal Processing we often need to look at relationships between real and imaginary parts of a complex signal. These relationships are generally described by Hilbert transforms. Hilbert transform not only helps us to relate the I and Q components but it is also used to create a special class of causal signals called analytic which are especially important in simulation. The analytic signals help us to represent band pass signals as complex signals which have specially attractive properties for signal processing. Hilbert Transform has other interesting properties.

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New One Step Iteration Schemes for two set-valued nonexpansive mappings

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Abstract: In this paper, we study a one step iteration scheme for two set-valued nonexpansive mappings and utilize the same to prove weak as well as strong convergence theorems. Thus, our results generalize and improve several relevant results contained in Abbas et al. (Appl. Math. Lett. 24 (2011), no. 2, 97-102), Khan (Bull. Belg. Math. Soc. Simon Stevin 17 (2010), 127-140), Khan (Nonlinear Anal. 8 (2005), 1295-1301), Fukhar-ud-din (J. Math. Anal. Appl. 328 (2007), 821-829) and Izhar Uddin et al. (Mediterr. J. Math. 13 (2016), 1211-1225).
Keywords: Banach space, Fixed point, Weak-convergence and Opial's property.
AMS Subject Classification: 54H25, 47H10.

1. Introduction

Multi-valued fixed point theory has applications in diverse domain such as: control theory, convex optimization, differential inclusion and economics (see [1] and references cited therein). The first existence of fixed points for multi-valued nonexpansive mappings in uniformly convex Banach spaces was due to Lim [2] while approximating fixed points results in respect of set-valued nonexpansive mappings due to Sastry and Babu [3] in setting of Hilbert space. Panyanak [4], Song and Wang [5] and Shahzad and Zegeye [6] iterative processes have been used to approximate the fixed point of nonexpansive multivalued nonexpansive mapping in Banach spaces. The approximating of common fixed points has its own importance as it has a direct application to optimization problem. Recently Abbas et al. [7] introduced a new one-step iteration scheme which is relatively simpler as well as natural in computation than the earlier schemes of Mann and Ishikawa.

In the paper of Abbas et al. [7], Izhar Uddin et al. [8] found some gapes and modified their iteration scheme and utilized the same to prove some convergence theorems in CAT (0) space. We also enlarge the class of all compact subsets (denoted by $C(K)$) of a closed convex subset K of a Banach space X by considering the relatively larger class of all proximinal subsets (denoted by $(P(K))$ of a closed convex subset K of X.

2. Preliminaries

Let X be a Banach space and K be a nonempty subset of X. Let $CB(K)$ be the family of nonempty closed bounded subsets of K while $C(K)$ be the family of nonempty compact subsets of K. A subset K of X is called proximinal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x,k) = d(x,K) = \inf\{\| x - y \| : y \in K\}.$$

It is well known that every closed convex subset of a uniformly convex Banach space is proximinal. In what follows, we denote by $PB(K)$, the family of nonempty bounded proximinal subsets of K. The Hausdorff metric H on $CB(K)$ is defined as

$$H(A,B) = max\{\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\}\, for\ A,B \in CB(K).$$

A set-valued mapping $T: K \to CB(K)$ is said to be nonexpansive if

$$H(T(x),T(y)) \leq \| x - y \|, for\ \ all\ x,y \in K.$$

A point $x \in K$ is said to be a fixed point of a set-valued T if $x \in Tx$. We employ the notation of $F(T)$ for the set of common fixed points of the mapping T while $F(S,T)$ stands for common fixed points of S and T.

A Banach space X is said to satisfy the Opial's condition if for any sequence $\{x_n\}$ in X with $x_n x$ (denotes weak convergence) implies that $\limsup_{n\to\infty} \| x_n - x \| < \limsup_{n\to\infty} \| x_n - y \|$ for all $y \in X$ with $y \neq x$.

The following two lemmas will be utilized to prove our results.

Lemma 2.1 [9]. Let X be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_{n\to\infty} \| x_n \| \leq r$, $\limsup_{n\to\infty} \| y_n \| \leq r$ and $\lim_{n\to\infty} \| t_n x_n + (1-t_n) y_n \| = r$ hold for some $r \geq 0$. Then

$$\lim_{n\to\infty} \| x_n - y_n \| = 0.$$

Lemma 2.2 [10] If $A,B \in CB(X)$ and $a \in A$, then for any $\varepsilon > 0$ there exists $b \in B$ such that

$$d(a,b) \leq H(A,B) + \varepsilon.$$

In this paper, we use the following iteration scheme which gives analogue of Izhar Uddin et al. [8] in Banach spaces. Construction of iteration process runs as follows: Let K be the closed and convex subset of a Banach space X. $S,T: K \to CB(K)$, where $CB(K)$ is the collection of all closed and bounded subsets of K. Now, choose $x_1 \in K$. In view of Lemma 2.2, for $p \in F(S,T)$, $\gamma_1 > 0$ and $\eta_1 > 0$ there exist $y_1 \in Sx_1$ and $z_1 \in Tx_1$ such that

$$\| y_1 - p \| \leq H(Sx_1, Sp) + \gamma_1$$

and

$$\| z_1 - p \| \leq H(Tx_1, Tp) + \eta_1.$$

Write,

$$x_2 = \alpha_1 x_1 + \beta_1 y_1 + \gamma_1 z_1$$

with

$$\alpha_1 + \beta_1 + \gamma_1 = 1.$$

For $p \in F(S,T)$, $\gamma_2 > 0$ and $\eta_2 > 0$, owing to Lemma 2.2, we can choose $y_2 \in Sx_2$ and $z_2 \in Tx_2$ satisfying

$$\| y_2 - p \| \leq H(Sx_2, Sp) + \gamma_2$$

and

$$\| z_2 - p \| \leq H(Tx_2, Tp) + \eta_2.$$

Define

$$x_3 = \alpha_2 x_2 + \beta_2 y_2 + \gamma_2 z_2$$

with

$$\alpha_2 + \beta_2 + \gamma_2 = 1.$$

Inductively, we can write

$$x_{n+1} = \alpha_n x_n + \beta_n y_n + \gamma_n z_n \tag{2.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ while $y_n \in Sx_n$ and $z_n \in Tx_n$ satisfy

$$\| y_n - p \| \leq H(Sx_n, Sp) + \gamma_n$$

and

$$\| z_n - p \| \leq H(Tx_n, Tp) + \eta_n$$

with $\lim_{n\to\infty} \gamma_n = 0$ and $\lim_{n\to\infty} \eta_n = 0$ for $p \in F(S,T)$.

3. Weak and strong convergence theorems

We begin with the following lemma.

Lemma 3.1 Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $S, T: K \to CB(K)$ be two set-valued nonexpansive mappings with $F(S,T) \neq \emptyset$ such that $Sp = \{p\} = Tp$ for all $p \in F(S,T)$. If the sequence $\{x_n\}$ is described by (2.3), then $\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n)$.

Proof. Let $p \in F(S,T)$. Consider,

$$\begin{aligned} \| x_{n+1} - p \| &= \| \alpha_n x_n + \beta_n y_n + \gamma_n z_n - p \| \\ &= \| \alpha_n (x_n - p) + \beta_n (y_n - p) + \gamma_n (z_n - p) \| \\ &\leq \alpha_n \| x_n - p \| + \beta_n H(Sx_n, Sp) + \gamma_n H(Tx_n, Tp) \\ &\leq \alpha_n \| x_n - p \| + \beta_n \| x_n - p \| + \gamma_n \| x_n - p \| = \| x_n - p \|, \end{aligned} \tag{3.1}$$

which amounts to say that for each $p \in F(S,T)$, sequence $\| x_n - p \|$ is a decreasing sequence of reals so that $\lim_{n\to\infty} \| x_n - p \|$ exists. We suppose that $\lim_{n\to\infty} \| x_n - p \| = c$ for some $c \geq 0$. Now, consider

$$\begin{aligned} \| x_{n+1} - p \| &= \| \alpha_n x_n + \beta_n y_n + \gamma_n z_n - p \| \\ &= \| \alpha_n (x_n - p) + \beta_n (y_n - p) + \gamma_n (z_n - p) \| \\ &= \| \alpha_n (x_n - p) + (1 - \alpha_n) \{ \tfrac{\beta_n}{1-\alpha_n} (y_n - p) + \tfrac{\gamma_n}{1-\alpha_n} (z_n - p) \} \| \end{aligned}$$

so that

$$\lim_{n\to\infty} \| \alpha_n (x_n - p) + (1 - \alpha_n) \left\{ \frac{\beta_n}{1-\alpha_n} (y_n - p) + \frac{\gamma_n}{1-\alpha_n} (z_n - p) \right\} \| = \lim_{n\to\infty} \| x_{n+1} - p \| = c. \tag{3.2}$$

On the other hand, for each $p \in F(S,T)$ we have

$$\begin{aligned} \| y_n - p \| &\leq H(Sx_n, Sp) + \gamma_n \\ &\leq \| x_n - p \| + \gamma_n \end{aligned}$$

which on taking $\limsup_{n\to\infty}$ yields that

$$\limsup_{n\to\infty} \| y_n - p \| \leq c.$$

Similarly, one can infer

$$\limsup_{n\to\infty} \| z_n - p \| \leq c.$$

Next consider,

$$\begin{aligned}\| \tfrac{\beta_n}{1-\alpha_n}(y_n - p) + \tfrac{\gamma_n}{1-\alpha_n}(z_n - p) \| &\leq \tfrac{\beta_n}{1-\alpha_n} \| y_n - p \| + \tfrac{\gamma_n}{1-\alpha_n} \| z_n - p \| \\ &= \tfrac{\beta_n}{1-\alpha_n} d(y_n, Sp) + \tfrac{\gamma_n}{1-\alpha_n} d(z_n, Tp) \\ &\leq \tfrac{\beta_n}{1-\alpha_n} H(Sx_n, Sp) + \tfrac{\gamma_n}{1-\alpha_n} H(Tx_n, Tp) \\ &\leq \tfrac{\beta_n}{1-\alpha_n} \| x_n - p \| + \tfrac{\gamma_n}{1-\alpha_n} \| x_n - p \| \\ &= \tfrac{\beta_n+\gamma_n}{1-\alpha_n} \| x_n - p \| = \| x_n - p \|,\end{aligned}$$

which on letting $n \to \infty$, reduces to

$$\limsup_{n\to\infty} \| \tfrac{\beta_n}{1-\alpha_n}(y_n - p) + \tfrac{\gamma_n}{1-\alpha_n}(z_n - p) \| \leq c. \tag{3.3}$$

In view of Lemma 2.1, Equation 3.2 and Equation 3.3, we have

$$\lim_{n\to\infty} \| (x_n - p) - \tfrac{\beta_n}{1-\alpha_n}(y_n - p) + \tfrac{\gamma_n}{1-\alpha_n}(z_n - p) \| = 0.$$

or

$$\lim_{n\to\infty} \tfrac{1}{1-\alpha_n} \| (1-\alpha_n)(x_n - p) - \beta_n(y_n - p) + \gamma_n(z_n - p) \| = 0,$$

or

$$\lim_{n\to\infty} \tfrac{1}{1-\alpha_n} \| x_n - p - (\alpha_n x_n + \beta_n y_n + \gamma_n z_n) + (\alpha_n + \beta_n + \gamma_n)p \| = 0$$

or

$$\lim_{n\to\infty} \tfrac{1}{1-\alpha_n} \| x_n - x_{n+1} \| = 0$$

yielding thereby

$$\lim_{n\to\infty} \| x_n - x_{n+1} \| = 0.$$

Similarly, we can also show that

$$\lim_{n\to\infty} \| y_n - x_{n+1} \| = 0$$

and

$$\lim_{n\to\infty} \| z_n - x_{n+1} \| = 0.$$

Owing to the facts $\| x_n - z_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - z_n \|$ and $\| x_n - y_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - y_n \|$, we have

$$\lim_{n\to\infty} \| x_n - z_n \| = 0$$

and

$$\lim_{n\to\infty} \| x_n - y_n \| = 0.$$

As $d(x_n, Sx_n) \leq \| x_n - y_n \|$, $d(x_n, Tx_n) \leq \| x_n - z_n \|$, on taking limit of both the sides, we get $\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n)$. This concludes the proof.

Theorem 3.2 Let K be a nonempty closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition $S, T: K \to C(K)$ be two set-valued nonexpansive mappings such that $F(S,T) \neq \emptyset$ with $Sp = \{p\} = Tp$ for all $p \in F(S,T)$. If sequence $\{x_n\}$ is described by (2.3), then $\{x_n\}$ converges weakly to a common fixed point of S and T.

Proof. From Lemma 3.1, we have $\lim_{n\to\infty} \| x_n - p \|$ exists for each $p \in F$ so that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0 = \lim_{n\to\infty} d(x_n, Sx_n)$. As, X is uniformly convex, there exist a subsequence x_{n_k} of x_n such that $x_{n_k}q$ for some $q \in K$. Now, we show that $q \in F(S,T)$. Suppose q does not belong to Tq. By the compactness of Tq, for any given x_{n_k}, there is $y_k \in Tq$ such that

$$\| x_{n_k} - y_k \| = d(x_{n_k}, Tq).$$

Again, owing to compactness of Tq there exists subsequence z_k of y_k such that $z_k \to z$ for some $z \in Tq$. Now, we show that $z = q$. If $z \neq q$, then we have

$$\begin{aligned}
\limsup_{n\to\infty} \| x_{n_k} - z \| &\leq \limsup_{n\to\infty}\{\| x_{n_k} - z_k \| + \| z_k - z \|\} \leq \limsup_{n\to\infty} \| x_{n_k} - z_k \| \\
&= \limsup_{n\to\infty} d(x_{n_k}, Tq) \\
&\leq \limsup_{n\to\infty}\{d(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tq)\} \\
&\leq \limsup_{n\to\infty} H(Tx_{n_k}, Tq) \leq \limsup_{n\to\infty} \| x_{n_k} - q \| \\
&< \limsup_{n\to\infty} \| x_{n_k} - z \|
\end{aligned}$$

which is a contradiction so that $z = q \in Tq$. Similarly, one can show that $q \in Sq$.

Now, we prove that $\{x_n\}$ has unique weak subsequential limit in $F(S,T)$. To show this, Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_i}z_1$ and $x_{n_j}z_2$. If $z_1 \neq z_2$, then owing to Opial's condition

$$\begin{aligned}
\lim_{n\to\infty} \| x_n - z_1 \| &= \lim_{i\to\infty} \| x_{n_i} - z_1 \| \\
&< \lim_{i\to\infty} \| x_{n_i} - z_2 \| \\
&= \lim_{n\to\infty} \| x_n - z_2 \| \\
&= \lim_{j\to\infty} \| x_{n_j} - z_2 \| \\
&< \lim_{j\to\infty} \| x_{n_j} - z_1 \| \\
&= \lim_{n\to\infty} \| x_n - z_1 \|
\end{aligned}$$

which is a contradiction and hence $\{x_n\}$ converges weakly to a common fixed point of S and T.

Theorem 3.3 Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $S,T: K \to CB(K)$ be two set-valued nonexpansive mappings such that $F(S,T) \neq \emptyset$ with $Sp = \{p\} = Tp$ for all $p \in F(S,T)$. If sequence $\{x_n\}$ is described by (2.3), then $\{x_n\}$ converges to a common fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$.

Proof. It is very easy to see that if x_n converges to a point $x \in F(S,T)$, then $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$.

To establish the converse part, suppose that $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$. By Equation (3.1), $for\ \ any\, p \in F(S,T)$ we have

$$\| x_{n+1} - p \| \leq \| x_n - p \|$$

so that

$$d(x_{n+1}, F(S,T)) \leq d(x_n, F(S,T)),$$

ensuring the existence of $\lim_{n\to\infty} d(x_n, F(S,T))$. As, $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$ so that $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in K. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$, there exists n_0 such that for all $n \geq n_0$, we have

$$d(x_n, F(S,T)) < \frac{\varepsilon}{4}.$$

In particular,

$$\inf\{\| x_{n_0} - p \| : p \in F(S,T)\} < \frac{\varepsilon}{4},$$

so there must exist a $p \in F(S,T)$ such that

$$\| x_{n_0} - p \| < \frac{\varepsilon}{2}.$$

Now for $m, n \geq n_0$, we have

$$\| x_{n+m} - x_n \| \leq \| x_{n+m} - p \| + \| x_n - p \| < 2 \| x_{n_0} - p \| < 2\frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space X and therefore it must converge in K. Let $\liminf_{n\to\infty} x_n = q$.

Now

$$\begin{aligned} d(q, Tq) &\leq \| q - x_n \| + d(x_n, Tx_n) + H(Tx_n, Tq) \\ &\leq \| q - x_n \| + \| x_n - z_n \| + \| x_n - q \| \to 0 as n \to \infty \end{aligned}$$

and hence $q \in Tq$. Similarly, we can also show that $q \in Sq$. For if,

$$\begin{aligned} d(q, Sq) &\leq \| q - x_n \| + d(x_n, Sx_n) + H(Sx_n, Sq) \\ &\leq \| q - x_n \| + \| x_n - z_n \| + \| x_n - q \| \to 0 as n \to \infty \end{aligned}$$

which implies that $q \in Sq$ and hence $q \in F(S,T)$.

Khan and Fukhar-ud-din [12] introduced the analogue of condition (A) for two mappings and gave an improved version in [13]. A set-valued version of condition (A) which is weaker than compactness of the domain,is given as follows. Two set-valued nonexpansive mappings $S, T: K \to CB(K)$ are said to satisfy condition (A) if there exists a nondecreasing function $f: [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $d(x, Tx) \geq f(d(x, F))$ or $d(x, Sx) \geq f(d(x, F))$ for all $x \in K$.

Now, we prove strong convergence theorem by using Condition (A) which extend the Theorem 2 of [5] and Theorem 3.3(b) of [11].

Theorem 3.4 Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $S, T: K \to CB(K)$ be two set-valued nonexpansive mappings which satisfy condition (A) with $F(S,T) \neq \emptyset$. If sequence $\{x_n\}$ is described by (2.3), the $\{x_n\}$ converges strongly to a common fixed point of S and T.

Proof. As in Theorem 3.3 $\lim_{n\to\infty} d(x_n, F(S,T))$ exists. Now, owing to condition (A), either

$$\lim_{n\to\infty} f(d(x_n, F(S,T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, F(S,T))) \leq \lim_{n\to\infty} d(x_n, Sx_n) = 0,$$

which amounts to say that $\lim_{n\to\infty} f(d(x_n, F(S,T))) = 0$. Since f is a nondecreasing function and $f(0) = 0$, $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$. Now, in view of Theorem 3.3, result follows.

4. convergence theorems without end point condition

With view to remove end point condition, we introduced the following iteration schemes.

Let K be the closed and convex subset of a Banach space X. $S, T: K \to P(K)$, where $P(K)$ is the collection of proximinal subsets of K. Here it can be pointed out that we consider the collection of all proximinal subsets of Banach space X which properly contains the collection of all compact subsets of X.

Choose $x_1 \in K$. As Sx_1 and Tx_1 are the proximinal subsets of K, we can choose points $y_1 \in Sx_1$ and $z_1 \in Tx_1$ such that $d(p, y_1) = d(p, Sx_1)$ and $d(p, z_1) \leq d(p, Tx_1)$ for any $p \in F(S,T) := F(S) \cap F(T)$.
Next, defined x_2 as follows:

$$x_2 = \alpha_1 x_1 + \beta_1 y_1 + \gamma_1 z_1$$

where α_1, β_1 and γ_1 lies in (0,1) such that $\alpha_1 + \beta_1 + \gamma_1 = 1$.

Due to proximinliaty of Sx_2 and Tx_2, again we can choose points $y_2 \in Sx_2$ and $z_2 \in Tx_2$ such that $d(p, y_2) = d(p, Sx_2)$ and $d(p, z_2) = d(p, Tx_2)$ enabling us to define x_3 as

$$x_3 = \alpha_2 x_2 + \beta_2 y_2 + \gamma_2 z_2,$$

where α_1, β_1 and γ_1 lies in (0,1) such that $\alpha_2 + \beta_2 + \gamma_2 = 1$.

Recursively, we define x_{n+1} as:

$$x_{n+1} = \alpha_n x_n + \beta_n y_n + \gamma_n z_n (4.1)$$

where y_n and z_n are sequences of Sx_n and Tx_n such that $d(p, y_n) = d(p, Sx_n)$ and $d(p, z_n) = d(p, Tx_n)$ for $p \in F(S,T) =: F(S) \cap F(T)$ and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1) such that $\alpha_n + \beta_n + \gamma_n = 1$.
The following lemma is very important to prove our main results.
We begin with the following lemma.
Lemma 4.1 Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $S, T: K \to P(K)$ be two set-valued nonexpansive mappings with $F(S,T) \neq \emptyset$. If sequence $\{x_n\}$ is described by (4.1), then we have $\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n)$.
Proof. Let $p \in F(S,T)$ and consider,

$$\begin{aligned} \| x_{n+1} - p \| &= \| \alpha_n x_n + \beta_n y_n + \gamma_n z_n - p \| \\ &= \| \alpha_n (x_n - p) + \beta_n (y_n - p) + \gamma_n (z_n - p) \| \\ &\leq \alpha_n \| x_n - p \| + \beta_n \| y_n - p \| + \gamma_n \| z_n - p \| \\ &= \alpha_n \| x_n - p \| + \beta_n d(p, Sx_n) + \gamma_n d(p, Tx_n) \\ &\leq \alpha_n \| x_n - p \| + \beta_n H(Sp, Sx_n) + \gamma_n H(Tp, Tx_n) \\ &\leq \alpha_n \| x_n - p \| + \beta_n \| x_n - p \| + \gamma_n \| x_n - p \| = \| x_n - p \|, \quad (4.1) \end{aligned}$$

which amounts to say that for each $p \in F(S,T)$, sequence $\| x_n - p \|$ is a decreasing sequence of reals so that $\lim_{n\to\infty} \| x_n - p \|$ exists. We suppose that $\lim_{n\to\infty} \| x_n - p \| = c$ for some $c \geq 0$. Now, consider

$$\begin{aligned} \| x_{n+1} - p \| &= \| \alpha_n x_n + \beta_n y_n + \gamma_n z_n - p \| \\ &= \| \alpha_n (x_n - p) + \beta_n (y_n - p) + \gamma_n (z_n - p) \| \\ &= \| \alpha_n (x_n - p) + (1 - \alpha_n)\{\tfrac{\beta_n}{1-\alpha_n}(y_n - p) + \tfrac{\gamma_n}{1-\alpha_n}(z_n - p)\} \| \end{aligned}$$

so that

$$\lim_{n\to\infty} \| \alpha_n (x_n - p) + (1 - \alpha_n)\{\tfrac{\beta_n}{1-\alpha_n}(y_n - p) + \tfrac{\gamma_n}{1-\alpha_n}(z_n - p)\} \| = \lim_{n\to\infty} \| x_{n+1} - p \| = c. (4.2)$$

On the other hand, for each $p \in F(S,T)$ we have

$$\| y_n - p \| = d(p, Sx_n) \leq H(Sp, Sx_n) \leq \| x_n - p \|$$

which on making $n \to \infty$ yields that

$$\lim_{n\to\infty} \| y_n - p \| \leq c.$$

Similarly, one can infer

$$\lim_{n\to\infty} \| z_n - p \| \leq c.$$

Rest of proof is followed by Lemma 3.1.

Now, we prove the following weak convergence theorem:

Theorem 4.2 Let K be a nonempty closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition and $S,T: K \to P(K)$ be two set-valued nonexpansive mappings with $F(S,T) \neq \emptyset$. If sequence $\{x_n\}$ is described by (4.1), then $\{x_n\}$ converges weakly to a common fixed point of S and T.

Proof. In view of Lemma 4.1, we observe that the sequence $\{x_n\}$ is a bounded sequence. As X is uniformly convex, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converges weakly to w for some $w \in K$. Now, we claim that $w \in Sw$. As, Sw is proximinal subset of K, for each x_{n_k} in K there exists y_k in Sw such that

$$d(x_{n_k}, y_k) = d(x_{n_k}, Sw).$$

Let $\{z_k\}$ be the subsequence of y_k such that $z_k \to z$. Obviously, $z \in Sw$. Now, we claim that $z = w$. Let on contrary that $z \neq w$, then

$$\begin{aligned}
\limsup_{n\to\infty} \| x_{n_k} - z \| &\leq \limsup_{n\to\infty}\{\| x_{n_k} - z_k \| + \| z_k - z \|\} \leq \limsup_{n\to\infty} \| x_{n_k} - z_k \| \\
&= \limsup_{n\to\infty} \| x_{n_k} - Sw \| \\
&\leq \limsup_{n\to\infty}\{\| x_{n_k} - Sx_{n_k} \| + H(Sx_{n_k}, Sw)\} \\
&\leq \limsup_{n\to\infty} H(Sx_{n_k}, Sw) \leq \limsup_{n\to\infty} \| x_{n_k} - w \| \\
&< \limsup_{n\to\infty} \| x_{n_k} - z \|
\end{aligned}$$

which is a contradiction so that $z = w \in Sw$. Similarly, we can show that $w \in Tw$. Rest of the proof will follow on the line of the Theorem 3.2.

Now, we prove the following strong convergence theorem:

Theorem 4.3 Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $S,T: K \to P(K)$ be two set-valued nonexpansive mappings with $F(S,T) \neq \emptyset$. If sequence $\{x_n\}$ is described by (4.1), then $\{x_n\}$ converges to a common fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$.

Proof. Proof of the theorem is line by line same as of Theorem 3.3.

Now, we prove strong convergence theorem by using Condition (A) which extend the Theorem 2 of [5] and Theorem 3.3(b) of [11].

Theorem 4.4 Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $S,T: K \to P(K)$ be two set-valued nonexpansive mappings which satisfy condition (A) with $F(S,T) \neq \emptyset$. If sequence $\{x_n\}$ is described by (4.1), the $\{x_n\}$ converges to a common fixed point of S and T.

Proof. Proof is same as Theorem 3.4.

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Modified Szasz type operators via Sheffer polynomials

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Abstract
In this note, we prove the Voronovskaja type results for Szasz and Kantorovich Szasz-type operators including Sheffer polynomials.
Keywords:Szasz operators, Sheffer Polynomials, Voronovskaja.

1. Introduction

For two times differentiable functions, Voronovaskaja [8] was the first to prove a theorem for Bernstein Polynomials known as Voronovskaja type theorem. Later on, similar studies were carried out by Butzar and Nessel [2], Rempulska and Skorupka [5] for some other operators.First, we recall n^{th} Bernstein operators due to S. N. Bernstein [1] as follows:

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $f \in C[0,1]$ and $0 \leq x \leq 1$. The purpose of this probabilistic method was to prove Weierstass approximation theorem more elegantly. In 1950, Szasz [6] generalized this operator for unbounded interval on the space of continuous functions defined on $[0,\infty)$ as

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad \forall x \in [0,\infty),$$

where f is a continuous function on $[0,\infty)$ and $n \in N$.

A new type of generalization of Szasz-Mirakjan operators which involves Appell polynomials was given by Jakimovski and Leviation [4] as follows:

$$P_n(f;x) = \frac{e^{-nx}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right).$$

In above relation p_k are Appell polynomials defined by the generating functions

$$A(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x) u^k,$$

where $A(z) = \sum_{k=0}^{\infty} a_k z^k (a_0 = 0)$ is an analytic function in the disc $|z| < R (R > 1)$ and $A(z) \neq 0$.

A more generalized form of Szasz operators including Sheffer polynomials was given by Ismail [3]

$$T_n(f;x)=\frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty} p_k(nx)f(\tfrac{k}{n}). \tag{1}$$

In above relation p_k are Sheffer polynomials given by the generating functions

$$A(u)e^{xH(u)}=\sum_{k=0}^{\infty} p_k(x)u^k (2)$$

where

$$A(z)=\sum_{k=0}^{\infty} a_k z^k \quad (a_0\neq 0)$$
$$H(z)=\sum_{k=0}^{\infty} h_k z^k \quad (h_1\neq 0) \tag{3}$$

be analytic functions in the disc $|z|<R(R>1)$. Under the following restrictions:

(i) For $x\in[0,\infty)$ and $k\in N\cup 0$, $p_k(x)\geq 0$,

(ii) $A(1)\neq 0$ and $H'(1)=1$,

(iii) relation (2) is valid for $|u|<R$ and the power series given by (3) converges for $|z|<R$, $R>1$.

Moreover, Ismail introduced the Kantorovich form of the operator (1) as

$$T_n^*(f;x)=n\frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty} p_k(nx)\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(s)ds. \tag{4}$$

Recently, Sezgin Sucu and Ertan Ibikli [7] proved results on rate of convergence using modulus of continuity for (1) and (2). Motivated by the above development, we have used T_n and T_n^* to prove the Vorovovskaja type theorems in the present paper.

2. Some properties of the operator T_n

We recall following lemmas due to Sezgin at al. [7]:

Lemma 2.1. Let $e_i=t^i$, $i=0,1,2$, $x\in[0,\infty)$, we have

$(i)\quad T_n(e_0;x)=1,$

$(ii)\quad T_n(e_1;x)=x+\frac{A\prime(1)}{nA(1)},$

$(iii)\quad T_n(e_2;x)=x^2+(\frac{2A\prime(1)}{A(1)}+H''(1)+1)\frac{x}{n}+\frac{A\prime(1)+A''(1)}{n^2A(1)}.$

Lemma 2.2. Let $\psi_x^i(t)=(t-x)^i, i=0,1,2$, for $x\geq 0$ and $n\in N$ we have

$(i)\quad T_n(\psi_x^0(t);x)=1,$

$(ii)\quad T_n(\psi_x^1(t);x)=\frac{A\prime(1)}{nA(1)},$

$(iii)\quad T_n(\psi_x^2(t);x)=(\frac{H\prime\prime(1)+1}{n})+\frac{A\prime(1)+A\prime\prime(1)}{n^2A(1)}.$

Next we prove

Lemma 2.3. For $x\geq 0$, we have

$$(i)\quad T_n(e_3;x)=x^3+\left(3+\frac{3A'(1)}{A(1)}+3H''(1)\right)\frac{x^2}{n}$$
$$+\left(\frac{2+3A''(1)}{A(1)}+\frac{6A'(1)}{A(1)}+\frac{3A'(1)H''(1)}{A(1)}+H''(1)+H'''(1)\right)\frac{x}{n^2}$$
$$+\frac{2A\prime(1)+3A''(1)+A'''(1)}{n^3A(1)},$$

$$(ii)\quad T_n(e_4;x)=x^4+(6+\frac{4A'(1)}{A(1)}+6H''(1))\frac{x^3}{n}+(11+\frac{6A''(1)}{A(1)}+\frac{18A'(1)}{A(1)}+18H''(1)$$
$$+\frac{9A'(1)H''(1)}{A(1)}+3(H''(1))^2$$
$$+4H'''(1))\frac{x^2}{n^2}+(6+\frac{4A'''(1)}{A(1)}+\frac{18A''(1)}{A(1)}+\frac{22A'(1)}{A(1)}+\frac{6A''(1)H''(1)}{A(1)}+\frac{18A'(1)H''(1)}{A(1)}$$
$$+\frac{4A'(1)H'''(1)}{A(1)}+6H'''(1)+11H''(1)+H''''(1))\frac{x}{n^3}+\frac{6A'(1)+11A''(1)+A''''(1)}{A(1)}.$$

Proof. From the generating functions of Sheffer polynomials, we obtain

$\sum_{K=0}^{\infty} K^3P_K(nx)=[(2A'(1)+3A''(1)+A'''(1))+nx(3A''(1)+6A'(1)+3A'(1)H''(1)+3A(1)H''(1)+2A(1)+A(1)H'''(1))+n^2 2x^2(3A(1)+3A'(1)+3A(1)H''(1))+n^3x^3A(1)]e^nxH(1),$

$\sum_{K=0}^{\infty} K^4P_K(nx)=[(6A'(1)+11A''(1)+6A'''(1)+A''''(1))+nx(4A'''(1)+18A''(1)+22A'(1)+6A''(1)H''(1)+18A'(1)H''(1)+4A'(1)H'''(1)+6A(1)H'''(1)+11A(1)H''(1)$

$+6A(1)+A(1)H'''')+n^2x^2(11A(1)+18A'(1)+18A(1)H''(1)+6A''(1)+9A'(1)H''(1)$
$+3A(1)(H''(1))^2+4A(1)H'''(1))+n^3x^3(6A(1)+4A'(1)+6A(1)H''(1))$
$+n^4x^4A(1)]e^nxH(1).$

The proof of lemma is obvious using these relation.

Lemma 2.4. The operator (1) satisfies the following relation:

$$T_n(\psi_x^4(t);x)=(3+14H''(1)+\frac{3A'(1)H''(1)}{A(1)}+3(H'')^2+4H''(1))\frac{x^2}{n^2}+(6+\frac{6A''(1)}{A(1)}$$
$$+\frac{14A'(1)}{A(1)}+\frac{6A''(1)H''(1)}{A(1)}$$
$$+\frac{18A'(1)H''(1)}{A(1)}+\frac{4A''(1)H''(1)}{A(1)}+\frac{6A''(1)H''(1)}{A(1)}+6H'''(1)+11H''(1)$$
$$+H''''(1))\frac{x}{n^3}+\frac{6A'(1)+11A''(1)+A''''(1)}{n^4A(1)}.$$

Proof. Proof of this relation can be obtained using lemma 2.1 and linearity property of the operators

$$T_n((t-x)^4;x)=T_n(t^4;x)-4xT_n(t^3;x)+6x^2T_n(t^2;x)-4x^3T_n(t;x)+T_n(1;x).$$

Lemma 2.5. Let $x_0\in[0,\infty)$ be a fixed point and $g(t;x_0)\in C[0,\infty)$, such that

$$\lim_{t\to x_0} g(t;x_0)=0. \tag{5}$$

Then

$$\lim_{t\to x_0} T_n(g(t;x_0);0)=0.$$

Proof. Let $\varepsilon>0$. Then from (5) and properties of $g(t;x_0)$, there exists a positive number $\delta=\delta(\varepsilon)$ such that

$|g(t;x_0)|<\varepsilon$ whenever $|t-x|<\delta$ and $|g(t;x_0)|<M\forall t\in[0,\infty)$ and $n\in N$, we have

$$|T_n(g(t;x_0));x_0|\le\frac{e^{nxH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k(nx)|g(t;x_0)|$$
$$=\frac{e^{nxH(1)}}{A(1)}\sum_{|\frac{k}{n}-x_0|<\delta}p_k(nx)|g(t;x_0)|+\frac{e^{nxH(1)}}{A(1)}\sum_{|\frac{k}{n}-x_0|\ge\delta}p_k(nx)|g(t;x_0)|$$

$$\leq \frac{\varepsilon}{2}+\frac{M}{\delta^2}\frac{e^{nxH(1)}}{A(1)}\sum_{k=0}^{\infty}\left(\frac{k}{n}-x_0\right)^2 P_k(nx)$$
$$=\frac{\varepsilon}{2}+\frac{M}{n\delta^2}((H''(1)+1)x+\frac{A\prime(1)+A\prime\prime(1)}{nA(1)}).$$

For the fixed positive number ε, δ, M and $x_0 \geq 0$ there exists a natural number n_1, we have

$$\frac{M}{n\delta^2}\left((H''(1)+1)x+\frac{A'(1)+A''(1)}{nA(1)}\right)<\frac{\varepsilon}{2},\ n\geq n_1,$$

and the result follows.

3 . The Voronovskaya type theorem for T_n

Theorem 3.1. Let $f \in C^2[0,\infty)$. Then we have

$$\lim_{n\to\infty} n\{T_n(f;x)-f(x)\}=\frac{A'(1)}{A(1)}f'(x)+(H''(1)+1)x\frac{f''(x)}{2!},\quad \forall\, x\in[0,\infty).$$

Proof. Let $x_0 \in [0,\infty)$ be a fixed point. Then for $f \in C^2[0,\infty)$ and $t \in [0,\infty)$ we have by Taylor's formula

$$f(t)=f(x_0)+f'(x_0)(t-x_0)+\frac{1}{2}f''(x_0)(t-x)^2+\varphi(t;x_0)(t-x_0)^2$$

where $\varphi(t;x_0)\in C[0,\infty)$ and $\lim_{t\to x_0} g(t;x_0)=0$. Now, applying the operator on both the side and in the light of linearity property, we have

$$T_n(f;x)=f(x_0)T_n(1;x_0)+f'(x_0)T_n((t-x_0);x_0)+\frac{1}{2}f''(x_0)T_n((t-x)^2;x_0)$$
$$+T_n(\varphi(t;x_0)(t-x_0)^2;x_0)).$$

Substract $f(x_0)$ and then on multiplying by n both side, we obtain

$$n\{T_n(f;x_0)-f(x_0)\}=f'(x_0)nT_n((t-x_0);x_0)+\frac{f''(x_0)}{2}nT_n((t-x_0)^2;x_0)$$
$$+nT_n(\varphi(t;x_0)(t-x)^2;x_0).$$

We have

$$\lim_{n\to\infty} n\{T_n(f;x)-f(x)\}=\frac{A\prime(1)}{A(1)}f'(x)+(H''(1)+1)x\frac{f\prime\prime(x)}{2!}$$
$$+\lim_{n\to\infty} nT_n(\varphi(t;x_0)(t-x)^2;x_0).$$

Using Holder's inequality. The last term can be given by

$$nT_n(\varphi(t;x_0)(t-x)^2;x_0)\leq n^2T_n((t-x)^4;x_0)T_n(\varphi(t;x_0)^2;x_0).$$

Let $\eta(t;x_0)=\varphi^2(t;x_0)$. Then $\lim\eta(t;x_0)=\lim\varphi^2(t;x_0)=0$ as $n\to\infty$. By using lemma 2.5 and

$$\lim_{n\to\infty} n^2T_n(\psi_x^4(t);x)=(3+14H''(1)+\frac{3A'(1)H''(1)}{A(1)}+3(H''(1)^2)+4H''(1))x^2,$$

we get

$$\lim_{n\to\infty} nT_n(\varphi(t;x_0)(t-x)^2;x_0)=0,$$

which proves the theorem 3.1 .

4. Voronovskaya type theorem for T_n^*

The Voronovskaja type theorem for Kantorovich-Szasz type operators including Sheffer polynomials is given by

Theorem 4.4. Let $f \in C^2[0,\infty), x \geq 0$. Then we have

$$\lim_{n\to\infty} n\{T_n^*(f;x)-f(x)\} = (\frac{1}{2}+\frac{A\prime(1)}{A(1)})f'(x)+(H''(1)+1)x\frac{f\prime\prime(x)}{2!}.$$

Proof. Proof of this theorem is similar as theorem 3.1.

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Fixed point theorems using Rational inequalities and Expansive Mapping in Multiplicative Metric Space

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Abstract: The purpose of this paper is to establish some fixed point theorems for expansive mappings in Multiplicative metric space. In process some results are extended using rational inequalities.
Key words: multiplicative metric space, compatible mapping of various types, expansive mappings.

1. Introduction and Preliminaries

Fixed point theorems in metric spaces have significant applications. The betterment has been enormous since few decades. A new kind of space, called Multiplicative metric space was first introduced by Bashirov [1] in 2008 in which the ideal idea was that the general triangular inequality was interchanged by a 'multiplicative triangle inequality'.

In 1984, Wang et al. [7] published some interesting work on expansive mapping in metric spaces which correspond to some contractive mapping [11]. In 2009, Ahmed [12] established a common fixed point theorem for expansive mappings by using the concept of compatibility of type (A) in 2-metric space, the proved theorem by ahmed was the generalization of the result of Kang et al. [13] for expansive mappins also Daffer and Kaneko [10] used two self mappings to generalize the result of Wang et al. [7] in a complete metric space.

Definition 1.1. Let X be a nonempty set. A multiplicative metric is a mapping $\Omega : X \times X \to \mathbf{R}_+$ satisfying the following conditions:

(i) $\Omega(x,y) \geq 1 \ \forall\, x,y \in X$ and $\Omega(x,y) = 1$ iff $x = y$;

(ii) $\Omega(x,y) = \Omega(y,x) \ \forall\, x,y \in X$;

(iii) $\Omega(x,y) \leq \Omega(x,z).\Omega(z,y) \ \forall\, x,y \in X$ (multiplicative triangle inequality).

Then mapping *Ω* together with X , i.e. (X,Ω) is a multiplicative metric space. Some important topological properties of the relevant multiplicative metric space have discussed by Ozavsar and Cevikel [5].

Example 1.2. ([9]) Let $\Omega : \mathbf{R} \times \mathbf{R} \to [1,\infty)$ be defined as $\Omega(x,y) = a^{|x-y|}$, where $x,y \in \mathbf{R}$ and $a > 1$. Then *Ω* is a multiplicative metric and $(\mathbf{R},\Omega)$ is a multiplicative metric space. We may call it usual multiplicative metric spaces.

Example 1.3. ([9]) Let (X,d) be a metric space. Define a mapping Ω_a on X by

$$\Omega_a(x,y)=a^{d(x,y)}=\begin{cases}1 & \text{if } x=y,\\ a & \text{if } x\neq y.\end{cases}$$

where $x,y\in X$ and $a>1$. Then Ω_a is a multiplicative metric and (X,Ω_a) is known as the discrete multiplicative metric space.

Ozavsar and Cevikel [5], introduced Banach contraction principle for multiplicative metric space based on the definition of multiplicative contraction. Using the concept of multiplicative contraction, following fixed point theorem on complete multiplicative metric has proved.

Theorem 1.4. *Let f be a multiplicative contraction mapping of a complete multiplicative metric space* (X,Ω) *into itself. Then f has a unique fixed point.*

Some fixed point theorems for expansion mappings have discussed by Wang et al. [7] and in Rhoades [8] , which corresponds to some contractive mappings in metric spaces.

Definition 1.5. Let *f* be a self map in multiplicative metric space (X,Ω) . Then *f* is said to be

(1) *multiplicative contraction* if $\exists$ a real constant $q\in[0,1)$ such that

$$\Omega(fx,fy)\leq\Omega^{q}(x,y)\ \forall\, x,y\in X$$

(2) *expansive mapping* if $\exists$ a real constant $q>1$ such that

$$\Omega(fx,fy)\geq\Omega^{q}(x,y)\ \forall\, x,y\in X$$

Kang et al. [6] discussed the notion of compatible mappings and its variants as follows:

Definition 1.6. Let *f* and ϑ be two mappings of a multiplicative metric space (X,Ω) into itself. Then *f* and ϑ are called

(1) *Compatible* if

$$\lim_{n\to\infty}\Omega(f\vartheta x_n,\vartheta fx_n)=1,$$

where $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty}fx_n=\lim_{n\to\infty}\vartheta x_n=t \text{ for some } t\in X.$$

where $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty}fx_n=\lim_{n\to\infty}\vartheta x_n=t \text{ for some } t\in X .$$

(2) *Compatible of type (B)* if

$$\lim_{n\to\infty}\Omega(f\vartheta x_n,\vartheta\vartheta x_n)\leq[\lim_{n\to\infty}\Omega(f\vartheta x_n,\, ft).\lim_{n\to\infty}\Omega(ft,\, ffx_n)]^{1/2}$$

and

$$\lim_{n\to\infty}\Omega(\vartheta fx_n,\, ffx_n)\leq[\lim_{n\to\infty}\Omega(\vartheta fx_n,\vartheta t).\lim_{n\to\infty}\Omega(\vartheta t,\vartheta\vartheta x_n)]^{1/2},$$

where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}fx_n=\lim_{n\to\infty}gx_n=t$ for some $t\in X$.

(3) *Compatible of type (C)* if

$$\lim_{n\to\infty}\Omega(f\vartheta x_n,\vartheta\vartheta x_n)\le\left[\lim_{n\to\infty}\Omega(f\vartheta x_n,\, ft).\lim_{n\to\infty}\Omega(ft,\, ffx_n).\lim_{n\to\infty}\Omega(ft,\vartheta\vartheta x_n)\right]^{1/3}$$

and

$$\lim_{n\to\infty}\Omega(\vartheta fx_n,\, ffx_n)\le\left[\lim_{n\to\infty}\Omega(\vartheta fx_n,\vartheta t).\lim_{n\to\infty}\Omega(\vartheta t,\vartheta\vartheta x_n).\lim_{n\to\infty}\Omega(\vartheta t,\, ffx_n)]\right]^{1/3}$$

where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n=\lim_{n\to\infty}\vartheta x_n=t$ for some $t\in X$.

(4) *Compatible of type (P)* if

$$\lim_{n\to\infty}\Omega(ffx_n,\vartheta\vartheta x_n)=1,$$

where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n=\lim_{n\to\infty}\vartheta x_n=t$ for some $t\in X$.

Next is a result which is useful for our main results.

Proposition 1.7. *Let f and ϑ be compatible mappings of a multiplicative metric space (X,Ω) into itself. Suppose that $\lim_{n\to\infty} fx_n=\lim_{n\to\infty}\vartheta x_n=t$ for some $t\in X$. then $\lim_{n\to\infty}\vartheta fx_n=ft$ if f is continuous at t.*

2. Main Results

Theorem 2.1.[8] Let *f* be a mapping of a multiplicative metric space (X,Ω) into itself. Then *f* is said to be *multiplicative contraction* if $\exists$ a real constant $q\in(0,1]$ such that

$$\Omega(fx,\, fy)\le\Omega^q(x,y)\quad \forall\, x,y\in X.$$

Now we establish Theorem 2.1 in multiplicative metric spaces using rational inequality as follows

Theorem 2.2. *Let f and ϑ be compatible mappings of a complete multiplicative metric space into itself satisfying the condition*

$$\begin{bmatrix}\Omega(fx,\vartheta x)\Omega(fy,\vartheta x)\Omega(fx,\vartheta y).\left[\frac{\Omega(fx,fy)+\Omega(fx,\vartheta y)}{\Omega(\vartheta x,\vartheta y)+\Omega(fy,\vartheta x)}\right].\\ \left[\frac{2\Omega(fy,\vartheta x)}{\Omega(fy,\vartheta y)+\Omega(\vartheta x,\vartheta y)}\right].\left[\frac{\Omega(fx,\vartheta x)+\Omega(fx,\vartheta y)}{\Omega(\vartheta x,\vartheta y)+\Omega(fy,\vartheta x)}\right]\end{bmatrix}^{\frac{1}{4q}}\ge\ \Omega(\vartheta x,\vartheta y)$$

(1)

$\forall\, x,y\in X$, *where $q>1$ and assume that $\vartheta(X)\subset f(X)$ and f is continuous. Then f and ϑ have a unique common fixed point.*

Proof. Let $x_0\in X$. Since $\vartheta(X)\subset f(X)$, $\exists$ $x_1\in X$ such that $fx_1=\vartheta x_0$. In general, $\exists$ $x_{n+1}\in X$ such that

$$y_n=fx_{n+1}=\vartheta x_n$$

From (2.1), consider

$$\Omega(y_n,y_{n+1})=\Omega(\vartheta x_n,\vartheta x_{n+1})$$

$$\leq \left[\begin{array}{c} \Omega(fx_n,\vartheta x_n)\Omega(fx_{n+1},\vartheta x_n)\Omega(fx_n,\vartheta x_{n+1}). \\ \left[\frac{\Omega(fx_n,fx_{n+1})+\Omega(fx_n,\vartheta x_{n+1})}{\Omega(\vartheta x_n,\vartheta x_{n+1})+\Omega(fx_{n+1},\vartheta x_n)}\right]. \\ \left[\frac{2\Omega(fx_{n+1},\vartheta x_n)}{\Omega(fx_{n+1},\vartheta x_{n+1})+\Omega(\vartheta x_n,\vartheta x_{n+1})}\right].\left[\frac{\Omega(fx_n,\vartheta x_n)+\Omega(fx_n,\vartheta x_{n+1})}{\Omega(\vartheta x_n,\vartheta x_{n+1})+\Omega(fx_{n+1},\vartheta x_n)}\right]\end{array}\right]^{\frac{1}{4q}}$$

$$= \left[\begin{array}{c} \Omega(\vartheta x_{n-1},\vartheta x_n)\Omega(\vartheta x_n,\vartheta x_n)\Omega(\vartheta x_{n-1},\vartheta x_{n+1}). \\ \left[\frac{\Omega(\vartheta x_{n-1},\vartheta x_n)+\Omega(\vartheta x_{n-1},\vartheta x_{n+1})}{\Omega(\vartheta x_{n-1},\vartheta x_{n+1})+\Omega(\vartheta x_n,\vartheta x_n)}\right]. \\ \left[\frac{2\Omega(\vartheta x_n,\vartheta x_n)}{\Omega(\vartheta x_n,\vartheta x_{n+1})+\Omega(\vartheta x_n,\vartheta x_{n+1})}\right].\left[\frac{\Omega(\vartheta x_{n-1},\vartheta x_n)+\Omega(\vartheta x_{n-1},\vartheta x_{n+1})}{\Omega(\vartheta x_n,\vartheta x_{n+1})+\Omega(\vartheta x_n,\vartheta x_n)}\right]\end{array}\right]^{\frac{1}{4q}}$$

$$= \left[\begin{array}{c} \Omega(y_{n-1},y_n)\Omega(y_n,y_n)\Omega(y_{n-1},y_{n+1}). \\ \left[\frac{\Omega(y_{n-1},y_n)+\Omega(y_{n-1},y_{n+1})}{\Omega(y_n,y_{n+1})+\Omega(y_n,y_n)}\right]. \\ \left[\frac{2\Omega(y_n,y_n)}{\Omega(y_n,y_{n+1})+\Omega(y_n,y_{n+1})}\right].\left[\frac{\Omega(y_{n-1},y_n)+\Omega(y_{n-1},y_{n+1})}{\Omega(y_n,y_{n+1})+\Omega(y_n,y_n)}\right]\end{array}\right]^{\frac{1}{4q}}$$

$$= [\Omega^4(y_{n-1},y_n)]^{\frac{1}{4q}}$$

$$\Omega(\vartheta x_n,\vartheta x_{n+1}) = \Omega^{\frac{1}{q}}(y_{n-1},y_n)$$

$$\Omega(y_n,y_{n+1}) \leq [\Omega(y_{n-1},y_n)]^{\frac{1}{q}}$$

$$= \Omega^{1/q}(fx_n,fx_{n+1}) \leq \Omega^{1/q}(\vartheta x_n,\vartheta x_{n+1}) \leq \Omega^{1/q^2}(fx_{n-1},fx_n)$$

$$= \Omega^{1/q^2}(y_{n-2},y_{n-1})$$

In general, we have

$$\Omega(y_n,y_{n+1}) \leq \Omega^{k^n}(y_0,y_1)$$

where $k=\dfrac{1}{q}<1$ or $q>1.$

Now for $m,n\in N$ with $n<m,$ consider

$$\begin{aligned}\Omega(y_n,y_m) &\leq \Omega(y_n,y_{n+1}).\Omega(y_{n+1},y_{n+2})\ldots\,\Omega(y_{m-1},y_m)\\ &\leq \Omega^{k^n}(y_0,y_1).\Omega^{k^{n+1}}(y_0,y_1)\ldots\Omega^{k^{m-1}}(y_0,y_1)\\ &\leq \Omega^{\frac{k^n}{1-k}}(y_0,y_1)\end{aligned}$$

which approaches to 1 as $n\to\infty.$ It follows that the sequence $\{y_n\}$ is a multiplicative Cauchy sequence. Since (X,Ω) is complete, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty}\vartheta x_n = z.$$

Since f and ϑ are compatible and f is continuous, by Proposition 1.7,

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} f\vartheta_n = \lim_{n\to\infty}\vartheta fx_n = fz.$$

Consider

$$\Omega(\vartheta f x_n,\vartheta z) \le \left[\begin{array}{c} \Omega(ffx_n,\vartheta f x_n)\Omega(fz,\vartheta f x_n)\Omega(ffx_n,\vartheta z). \\ \left[\frac{\Omega(ffx_n,fz)+\Omega(ffx_n,\vartheta z)}{\Omega(\vartheta f x_n,\vartheta z)+\Omega(fz,\vartheta f x_n)}\right]. \\ \left[\left[\frac{2\Omega(fz,\vartheta f x_n)}{\Omega(fz,\vartheta z)+\Omega(\vartheta f x_n,\vartheta z)}\right].\left[\frac{\Omega(ffx_n,\vartheta f x_n)+\Omega(ffx_n,\vartheta z)}{\Omega(\vartheta f x_n,\vartheta z)+\Omega(fz,\vartheta f x_n)}\right]\right]\end{array}\right]^{\frac{1}{4q}}$$

Letting $n\to\infty$, we get

$$\lim_{n\to\infty}\vartheta fx_n = \vartheta z = fz.$$

Now it is to show that z is a fixed point of f and ϑ. Again consideringg

$$\Omega(\vartheta z,\vartheta x_n) \le \left[\begin{array}{c} \Omega(fz,\vartheta z)\Omega(fx_n,\vartheta z)\Omega(fz,\vartheta x_n). \\ \left[\frac{\Omega(fz,fx_n)+\Omega(fz,\vartheta x_n)}{\Omega(\vartheta z,\vartheta x_n)+\Omega(fx_n,\vartheta z)}\right]. \\ \left[\left[\frac{2\Omega(fx_n,\vartheta z)}{\Omega(fx_n,\vartheta x_n)+\Omega(\vartheta z,\vartheta x_n)}\right].\left[\frac{\Omega(fz,\vartheta z)+\Omega(fz,\vartheta x_n)}{\Omega(\vartheta z,\vartheta x_n)+\Omega(fx_n,\vartheta z)}\right]\right]\end{array}\right]^{\frac{1}{4q}}$$

Letting *n* approaches to infinity using the fact that $fz = \vartheta z$, we have

$$\Omega(\vartheta z,z) \le \left[\begin{array}{c} \Omega(\vartheta z,\vartheta z)\Omega(z,\vartheta z)\Omega(\vartheta z,z). \\ \left[\frac{\Omega(\vartheta z,z)+\Omega(\vartheta z,z)}{\Omega(\vartheta z,z)+\Omega(z,\vartheta z)}\right]. \\ \left[\left[\frac{2\Omega(z,\vartheta z)}{\Omega(z,z)+\Omega(\vartheta z,z)}\right].\left[\frac{\Omega(\vartheta z,\vartheta z)+\Omega(\vartheta z,z)}{\Omega(\vartheta z,z)+\Omega(z,\vartheta z)}\right]\right]\end{array}\right]^{\frac{1}{4q}}$$

$$\Omega(\vartheta z,z) \le [\Omega(z,\vartheta z)\Omega(\vartheta z,z).]^{\frac{1}{4q}}$$

which implies that $fz = \vartheta z = z$, since $q > 1$. And hence z is a fixed point of *f* and ϑ.

Uniqueness

Let z and w be the two distinct fixed points then we have,

$$\left[\begin{array}{c} \Omega(fz,\vartheta z)\Omega(fw,\vartheta z)\Omega(fz,\vartheta w).\left[\frac{\Omega(fz,fw)+\Omega(fz,\vartheta w)}{\Omega(\vartheta z,\vartheta w)+\Omega(fw,\vartheta z)}\right]. \\ \left[\frac{2\Omega(fw,\vartheta z)}{\Omega(fw,\vartheta w)+\Omega(\vartheta z,\vartheta w)}\right].\left[\frac{\Omega(fz,\vartheta z)+\Omega(fz,\vartheta w)}{\Omega(\vartheta z,\vartheta w)+\Omega(fw,\vartheta z)}\right]\end{array}\right]^{\frac{1}{4q}} \ge \Omega(\vartheta z,\vartheta w)$$

$$\left[\begin{array}{c} \Omega(z,z)\Omega(w,z)\Omega(z,w).\left[\frac{\Omega(z,w)+\Omega(z,w)}{\Omega(z,w)+\Omega(w,z)}\right]. \\ \left[\frac{2\Omega(w,z)}{\Omega(w,w)+\Omega(z,w)}\right].\left[\frac{\Omega(z,z)+\Omega(z,w)}{\Omega(z,w)+\Omega(w,z)}\right]\end{array}\right]^{\frac{1}{4q}} \ge \Omega(z,w)$$

$$[\Omega(w,z)\Omega(z,w)]^{\frac{1}{4q}} \ge \Omega(z,w).$$

Next we prove a common fixed point theorem for compatible mappings of type (B) as follows:

Theorem 2.3. *Let f and ϑ be compatible mappings of type (B) of a complete multiplicative metric space into itself satisfying the condition (1) and assume that $\vartheta(X) \subset f(X)$ and f is continuous. Then f and ϑ have a unique common fixed point.*

Proof. From the proof of Theorem 2.2, $\{\vartheta x_n\}$ is a multiplicative Cauchy sequence. Since (X,Ω) is complete, there exists a point $z \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} \vartheta x_n = z,$$

Since f is continuous, we have

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} f\vartheta x_n = fz.$$

Since f and ϑ are compatible of type (B), so

$$\lim_{n\to\infty}\Omega(f\vartheta x_n, \vartheta\vartheta x_n) \le \left[\lim_{n\to\infty}\Omega(f\vartheta x_n, fz).\lim_{n\to\infty}\Omega(fz, ffx_n)\right]^{1/2}.$$

Letting $n\to\infty$, we have

$$\lim_{n\to\infty}\Omega(fz, \vartheta\vartheta x_n) \le \left[\Omega(fz, fz).\Omega(fz, fz)\right]^{1/2},$$

which implies that,

$$\lim_{n\to\infty}\vartheta\vartheta x_n = fz.$$

Consider

$$\Omega(\vartheta\vartheta x_n, \vartheta f x_n) \le \begin{bmatrix} \Omega(f\vartheta x_n, \vartheta\vartheta x_n)\Omega(ffx_n, \vartheta\vartheta x_n)\Omega(f\vartheta x_n, \vartheta f x_n). \\ \left[\frac{\Omega(f\vartheta x_n, ffx_n)+\Omega(f\vartheta x_n,\vartheta f x_n)}{\Omega(f\vartheta x_n,\vartheta f x_n)+\Omega(ffx_n,\vartheta\vartheta x_n)}\right]. \\ \left[\frac{2\Omega(ffx_n,\vartheta\vartheta x_n)}{\Omega(ffx_n,\vartheta f x_n)+\Omega(\vartheta\vartheta x_n,\vartheta f x_n)}\right].\left[\frac{\Omega(f\vartheta x_n,\vartheta\vartheta x_n)+\Omega(f\vartheta x_n,\vartheta f x_n)}{\Omega(ffx_n,\vartheta f x_n)+\Omega(ffx_n,\vartheta\vartheta x_n)}\right] \end{bmatrix}^{\frac{1}{4q}}$$

Taking limiting value of n approaching to infinity, we have

$$\le \begin{bmatrix} \Omega(fz, fz)\Omega(fz, fz)\Omega(fz, \vartheta f x_n). \\ \left[\frac{\Omega(fz,fz)+\Omega(fz,\vartheta f x_n)}{\Omega(\vartheta z,\vartheta f x_n)+\Omega(fz,fz)}\right]. \\ \left[\frac{2\Omega(fz,fz)}{\Omega(fz,\vartheta f x_n)+\Omega(fz,\vartheta f x_n)}\right].\left[\frac{\Omega(fz,fz)+\Omega(fz,\vartheta f x_n)}{\Omega(fz,\vartheta f x_n)+\Omega(fz,fz)}\right] \end{bmatrix}^{\frac{1}{4q}}$$

which implies that

$$\lim_{n\to\infty}\vartheta\vartheta x_n = \lim_{n\to\infty}\vartheta f x_n = fz$$

Again consider

$$\Omega(\vartheta f x_n, \vartheta z) \le \begin{bmatrix} \Omega(ffx_n, \vartheta f x_n)\Omega(fz, \vartheta f x_n)\Omega(ffx_n, \vartheta z). \\ \left[\frac{\Omega(ffx_n,fz)+\Omega(ffx_n,\vartheta z)}{\Omega(\vartheta f x_n,\vartheta z)+\Omega(fz,\vartheta f x_n)}\right]. \\ \left[\frac{2\Omega(fz,\vartheta f x_n)}{\Omega(fz,\vartheta z)+\Omega(\vartheta f x_n,\vartheta z)}\right].\left[\frac{\Omega(ffx_n,\vartheta f x_n)+\Omega(ffx_n,\vartheta z)}{\Omega(fz,\vartheta z)+\Omega(fz,\vartheta f x_n)}\right] \end{bmatrix}^{\frac{1}{4q}}$$

Letting $n\to\infty$, we have

$$\Omega(fz,\vartheta z) \leq \left[\begin{array}{c} \Omega(fz,fz)\Omega(fz,fz)\Omega(fz,\vartheta z). \\ \left[\frac{\Omega(fz,fz)+\Omega(fz,\vartheta z)}{\Omega(fz,\vartheta z)+\Omega(fz,fz)}\right]. \\ \left[\frac{2\Omega(fz,fz)}{\Omega(fz,\vartheta z)+\Omega(fz,\vartheta z)}\right].\left[\frac{\Omega(fz,fz)+\Omega(fz,\vartheta z)}{\Omega(fz,\vartheta z)+\Omega(fz,fz)}\right] \end{array}\right]^{\frac{1}{4q}}$$

$$\Omega(fz,\vartheta z) \leq [\Omega^2(fz,\vartheta z)]^{\frac{1}{4q}}$$

which implies that,

$$fz = \vartheta z.$$

Next we have to show that z is a fixed point of f and ϑ, that easily follows from the proof of Theorem 2.2.

Uniqueness follows from (1).This completes the proof.

Now we prove a common fixed point theorem for compatible mappings of type (C) as follows:

Theorem 2.4. *Let f and ϑ be compatible mappings of type (C) of a complete multiplicative metric space into itself satisfying the conditions (1) and assume that $\vartheta(X) \subset f(X)$ and f is continuous. Then f and Ω have a unique common fixed point.*

Proof. From the proof of Theorem 2.2, $\{\vartheta x_n\}$ is a multiplicative Cauchy sequence. Since (X,Ω) is complete, there exists a point $z \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} \vartheta x_n = z.$$

since f is continuous, we have

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} f\vartheta x_n = fz.$$

Since f and g are compatible of type (C), so

$$\lim_{n\to\infty}\Omega(f\vartheta x_n,\vartheta\vartheta x_n) \leq \left[\lim_{n\to\infty}\Omega(f\vartheta x_n, fz).\lim_{n\to\infty}\Omega(fz, ffx_n).\lim_{n\to\infty}\Omega(fz,\vartheta\vartheta x_n)\right]^{1/3}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty}\Omega(fz,\vartheta\vartheta x_n) \leq \left[\Omega(fz, fz).\Omega(fz, fz).\lim_{n\to\infty}\Omega(fz,\vartheta\vartheta x_n)\right]^{1/3},$$

which implies,

$$\lim_{n\to\infty} \vartheta\vartheta x_n = fz.$$

Consider

$$\lim_{n\to\infty}\Omega(\vartheta\vartheta x_n,\vartheta f x_n) \leq \left[\begin{array}{c} \Omega(f\vartheta x_n,\vartheta\vartheta x_n)\Omega(ffx_n,\vartheta\vartheta x_n)\Omega(f\vartheta x_n,\vartheta f x_n). \\ \left[\frac{\Omega(f\vartheta x_n,ffx_n)+\Omega(f\vartheta x_n,\vartheta f x_n)}{\Omega(\vartheta\vartheta x_n,\vartheta f x_n)+\Omega(ffx_n,\vartheta\vartheta x_n)}\right]. \\ \left[\frac{2\Omega(ffx_n,\vartheta\vartheta x_n)}{\Omega(ffx_n,\vartheta f x_n)+\Omega(\vartheta\vartheta x_n,\vartheta f x_n)}\right].\left[\frac{\Omega(f\vartheta x_n,\vartheta\vartheta x_n)+\Omega(f\vartheta x_n,\vartheta f x_n)}{\Omega(ffx_n,\vartheta f x_n)+\Omega(ffx_n,\vartheta\vartheta x_n)}\right] \end{array}\right]^{\frac{1}{4q}}$$

$$\leq \left[\begin{array}{c} \Omega(f\vartheta x_n,\vartheta\vartheta x_n)\Omega(ffx_n,\vartheta\vartheta x_n)\Omega(f\vartheta x_n,\vartheta f x_n). \\ \left[\frac{\Omega(f\vartheta x_n,ffx_n)+\Omega(f\vartheta x_n,\vartheta f x_n)}{\Omega(f\vartheta x_n,\vartheta f x_n)+\Omega(ffx_n,\vartheta\vartheta x_n)}\right]. \\ \left[\frac{2\Omega(ffx_n,\vartheta\vartheta x_n)}{\Omega(ffx_n,\vartheta f x_n)+\Omega(\vartheta\vartheta x_n,\vartheta f x_n)}\right].\left[\frac{\Omega(f\vartheta x_n,\vartheta\vartheta x_n)+\Omega(f\vartheta x_n,\vartheta f x_n)}{\Omega(ffx_n,\vartheta f x_n)+\Omega(ffx_n,\vartheta\vartheta x_n)}\right] \end{array}\right]^{\frac{1}{4q}}$$

which implies that

$$\lim_{n\to\infty} \vartheta\vartheta x_n = \lim_{n\to\infty} \vartheta f x_n = fz$$

Again consider

$$\Omega(\vartheta f x_n, \vartheta z) \le \begin{bmatrix} \Omega(ffx_n, \vartheta f x_n)\Omega(fz, \vartheta f x_n)\Omega(ffx_n, \vartheta z). \\ \left[\frac{\Omega(ffx_n,fz)+\Omega(ffx_n,\vartheta z)}{\Omega(ffx_n,\vartheta z)+\Omega(fz,\vartheta fx_n)}\right]. \\ \left[\frac{2\Omega(fz,\vartheta fx_n)}{\Omega(fz,\vartheta z)+\Omega(\vartheta fx_n,\vartheta z)}\right].\left[\frac{\Omega(ffx_n,\vartheta fx_n)+\Omega(ffx_n,\vartheta z)}{\Omega(fz,\vartheta z)+\Omega(fz,\vartheta fx_n)}\right] \end{bmatrix}^{\frac{1}{4q}}$$

Letting $n \to \infty,$ we have

$$\Omega(fz, \vartheta z) \le \begin{bmatrix} \Omega(fz, fz)\Omega(fz, fz)\Omega(fz, \vartheta z). \\ \left[\frac{\Omega(fz,fz)+\Omega(fz,\vartheta z)}{\Omega(fz,\vartheta z)+\Omega(fz,fz)}\right]. \\ \left[\frac{2\Omega(fz,fz)}{\Omega(fz,\vartheta z)+\Omega(fz,\vartheta z)}\right].\left[\frac{\Omega(fz,fz)+\Omega(fz,\vartheta z)}{\Omega(fz,\vartheta z)+\Omega(fz,fz)}\right] \end{bmatrix}^{\frac{1}{4q}}$$

Since $q > 1,$

$$fz = \vartheta z.$$

The rest of the proof follows easily from Theorem 2.2.

Now we prove a common fixed point theorem for compatible mappings of type (P) as follows:

Theorem 2.5. *Let f and ϑ be compatible mappings of type (P) of a complete metric space into itself satisfying the condition (1) and assume that $g(X) \subset f(X)$ and f is continuous. Then f and g have a unique common fixed point.*

Proof. From the proof of Theorem 2.2, $\{\vartheta x_n\}$ is a multiplicative Cauchy sequence. Since (X, Ω) is complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} \vartheta x_n = z.$$

Since *f* and *g* are compatible of type (P) and *f* is continuous, we have

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} \vartheta\vartheta x_n = \lim_{n\to\infty} f\vartheta x_n = fz.$$

Consider

$$\lim_{n\to\infty} \Omega(\vartheta\vartheta x_n, \vartheta z) \le \lim_{n\to\infty} \begin{bmatrix} \Omega(f\vartheta x_n, \vartheta\vartheta x_n)\Omega(fz, \vartheta\vartheta x_n)\Omega(f\vartheta x_n, \vartheta z). \\ \left[\frac{\Omega(f\vartheta x_n,fz)+\Omega(f\vartheta x_n,\vartheta z)}{\Omega(f\vartheta x_n,\vartheta z)+\Omega(fz,\vartheta\vartheta x_n)}\right]. \\ \left[\frac{2\Omega(fz,\vartheta\vartheta x_n)}{\Omega(fz,\vartheta z)+\Omega(\vartheta\vartheta x_n,\vartheta z)}\right].\left[\frac{\Omega(f\vartheta x_n,\vartheta\vartheta x_n)+\Omega(f\vartheta x_n,\vartheta z)}{\Omega(\vartheta\vartheta x_n,\vartheta z)+\Omega(fz,\vartheta\vartheta x_n)}\right] \end{bmatrix}^{\frac{1}{4q}}$$

$$\Omega(fz, \vartheta z) \le \begin{bmatrix} \Omega(fzfz)\Omega(fz, fz)\Omega(fz, \vartheta z). \\ \left[\frac{\Omega(fz,fz)+\Omega(fz,\vartheta z)}{\Omega(fz,\vartheta z)+\Omega(fz,fz)}\right]. \\ \left[\frac{2\Omega(fz,fz)}{\Omega(fz,\vartheta z)+\Omega(fz,\vartheta z)}\right].\left[\frac{\Omega(fz,fz)+\Omega(fz,\vartheta z)}{\Omega(fz,\vartheta z)+\Omega(fz,fz)}\right] \end{bmatrix}^{\frac{1}{4q}}$$

$$\Omega(fz,\vartheta z) \leq \Omega^{\frac{1}{2q}}(fzfz)$$

which implies that

$$\lim_{n\to\infty} \vartheta\vartheta x_n = \vartheta z = fz.$$

The rest of the proof follows easily from Theorem 2.2.

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Wavelet Frames on Local Fields of Positive Characteristic

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Abstract Wavelet frames have gained considerable popularity during the past decade, primarily due to their substantiated applications in diverse and widespread fields of engineering and science. In the present paper, we characterize the functions $\psi \in L^2(\mathbb{K})$, where, $\mathbb{K}$ is a local field, such that $\left\{q^{\frac{n}{2}}\psi(\mathfrak{p}^n(x-qu(m)))\right\}_{m,n\in\mathbb{N}_0}$ generates a frame for $L^2(\mathbb{K})$. In this concern, some results involving necessary and sufficient conditions are established.
Keywords: Fourier transform; Wavelet Frame; Local Field

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3. Introduction

Theory of frames, especially theory of the Gabor frames and wavelet frames ([3, 8, 7]), has a long history of the development even before the discovery of the multiresolution analysis of [9] and the systematic construction of compactly supported orthonormal wavelets of [4]. The concept of frame can be traced back to [5]. The wide scope of applications of frames can be found in the early literature on applications of Gabor and wavelet frames ([3, 8]). Such applications include time frequency analysis for signal processing, coherent state in quantum mechanics, filter bank design in electrical engineering, edge and singularity detection in image processing, and etc.

During the last decade, there is a tremendous interest in the problem of constructing wavelet bases and frames on various spaces other than $\mathbb{R}$, such as locally compact Abelian groups [6], Vilenkin groups [13], Cantor dyadic groups [11], p-adic fields [1] and zero-dimensional groups [1]. The local field $\mathbb{K}$ is a natural model for the structure of wavelet frame systems, as well as a domain upon which one can construct wavelet basis functions. There is a substantial body of work that has been concerned with the construction of wavelets on local fields or more generally on local fields of positive characteristic. For example, R. L. Benedetto and J. J. Benedetto [2] developed a wavelet theory for local fields and related groups. They did not develop the multi-resolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets.
In the present article, we discuss the characterization of wavelet frames on local fields of positive characteristic.
The paper is organized as follows. In Section 2, we discuss Fourier analysis of Local fields of positive characteristic and some results which are prerequisite in the subsequent chapters. Section 3 gives the complete characterization of the functions $\psi \in L^2(\mathbb{K})$, where $\mathbb{K}$ is a local field, so that

the system $\left\{p^{\frac{-n}{2}}\psi(\mathfrak{p}^{n}(x-qu(m)))\right\}_{m,n\in\mathbb{N}_0}$ forms a frame for $L^2(\mathbb{K})$. Some results in this concern are established.

4. Fourier Analysis on Local Fields

Any field $\mathbb{K}$ equipped with the discrete topology is called a Local field. If $\mathbb{K}$ is connected, then it is $\mathbb{R}$ or $\mathbb{C}$. However, if $\mathbb{K}$ is not connected, then it is totally disconnected. Thus a locally compact, indiscrete and totally disconnected field $\mathbb{K}$ is called a Local field. The additive and multiplicative groups of $\mathbb{K}$ are denoted by $\mathbb{K}^+$ and $\mathbb{K}^*$ respectively. Other than this, example of a Local field of characteristic zero is p-adic field $\mathbb{Q}_{\mathbb{p}}$, fields of positive characteristic are Cantor Dyadic group and Vilenkin p-groups. For further details we refer to [12, 10].

Let dx be the Haar measure on $\mathbb{K}^+$. If $\lambda \in \mathbb{K}\backslash 0$, the $d(\lambda x)$ is also a Haar measure on $\mathbb{K}^+$. If we let $d(\lambda x) = |\lambda| dx$, then we call $|\alpha|$ as the absolute or valuation of λ, which is non-Archimedian on $\mathbb{K}$. The valuation $x \to |x|$ with $|0| = 0$ has the following properties

(i) $|x| = 0$ if and only if $x = 0$,

(ii) $|xy| = |x||y|$ for all $x, y \in \mathbb{K}$,

(iii) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{K}$.

Property (iii) is called the ultra-metric inequality. Moreover, $|x+y| = \max\{|x|,|y|\}$, if $|x| \neq |y|$. Define $\mathcal{B} = \{x \in \mathbb{K}: |x| < 1\}$. Then the set $\mathcal{B}$ is called the prime ideal in $\mathbb{K}$ which is maximal ideal in $\mathcal{D} = \{x \in \mathbb{K}: |x| \leq 1\}$. Thus $\mathcal{B}$ is both principal and prime. Therefore for such an ideal $\mathcal{B}$ in $\mathcal{D}$, we have $\mathcal{B} = < p > = p\mathcal{D}$. Let $\mathcal{D}^* = \mathcal{D}\backslash\mathcal{B} = \{x \in \mathbb{K}: |x| = 1\}$. Then $\mathcal{D}^*$ is a group of units in K^* and if $x \neq 0$, then we write $x = p^k z, z \in \mathcal{D}^*$. Moreover, $\mathcal{B}^k = p^k\mathcal{D} = \left\{x \in \mathbb{K}: |x| < q^{-k}\right\}$ are compact subgroups of $\mathbb{K}^+$ and are known as the fractional ideals of $\mathbb{K}^+$. Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathcal{B}$ in $\mathcal{D}$, then any element $x \in \mathbb{K}$ can be uniquely written as $x = \sum_{r=k}^{\infty} c_r \mathbf{p}^r$, $c_r \in \mathcal{U}$. Let χ is constant on cosets of $\mathcal{D}$ so if $y \in \mathcal{B}^k$, then $\chi_y(x) = \chi(yx), x \in \mathbb{K}$. Suppose that χ_u is any character on K^+, then clearly the restriction $\chi_u|\mathcal{D}$ is also a character on $\mathcal{D}$ in K^+, as then, as it was proved in [12], the set $\left\{\chi_{u(n)}: n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$. The Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbb{K}) \cap L^2(\mathbb{K})$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\overline{\chi_\xi(x)}dx.$$

It is noted that

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)dx.$$

Furthermore, the properties of Fourier transform on local field $\mathbb{K}$ are much similar to those of on the real line. In particular Fourier transform is unitary on $L^2(\mathbb{K})$. For any prime p and $a, b \in \mathbb{K}$, let D_p, $T_{u(n)a}$ and $E_{u(m)b}$ be the operators acting on $L^2(\mathbb{K})$ given by dilations, translations and modulations, respectively:

$$D_p f(x) = \sqrt{q} f(p^{-1}x),$$
$$T_{u(n)a} f(x) = f(x - u(n)a),$$
$$E_{u(m)b} f(x) = \chi(u(m)bx) f(x).$$

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$, where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that span $\left\{\zeta_j\right\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying $0 \leq n < q, n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, 0 \leq a_k < p, and\ k = 0,1,\dots, c-1,$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\mathfrak{p}^{-1}.$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s, n \in \mathbb{N}_0, 0 \le b_k < q, k = 0,1,2,\ldots,s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}.$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But if $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, then $u(rq^k + s) = u(r)\mathbf{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k): k \in \mathbb{N}_0\} = \{u(k): k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Here after, we use the notation $\chi_n = \chi_{u(n)}, n \ge 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \neq 1. \end{cases}$$

We also denote the test function space on K by S, i.e., each function f in S is a finite linear combination of functions of the form $\mathbf{1_k}(x - h), h \in K, k \in \mathbb{Z}$, where $\mathbf{1}_k$ is the characteristic function of $\mathfrak{B}^k$. Then, it is clear that S is dense in $L^p(K), 1 \le p < \infty$, and each function in S is of compact support and so is its Fourier transform.

5. Main Results

Definition 3.1 Given a function ψ, we say $\psi \in W(L^\infty(\mathbb{K}), L^1(\mathbb{K}))$ for some $a > 0$, if

$$||\psi||_{W,\mathfrak{p}} = \sum_{n\in\mathbb{N}_0} ||\psi(x - \mathfrak{p}u(n))1_{\mathcal{B}^{-1}}||_\infty < \infty.$$

The functions $\psi \in W(L^\infty(\mathbb{K}), L^1(\mathbb{K}))$ have the following properties.

Lemma 3.2 (i) If $||\psi||_{W,\mathbf{p}}$ is finite for some $\mathbf{p}$, then it is finite for all $\mathbf{p}$.

(ii) if $0 < |\mathbf{p}| \le |q|$, then $||\psi||_{W,q} \le 2||\psi||_{W,\mathbf{p}}$

(iii) For any $\mathbf{p}, q$, we have

$$||\psi(x - qu(n))|| \le 2||\psi||_{W,\mathbf{p}}\,.$$

Proof. (i) Define $\mathcal{B}^k = \{x \in \mathbb{K}, |x| \le \mathbf{p^k}\}_{k\in\mathbb{N}_0}$ and $\mathcal{B}^r = \{x \in \mathbb{K}: |x| \le q^{-r}\}_{r\in\mathbb{N}_0}$. Let $\{\mathcal{B}^s\}_{s\in\mathbb{N}}$ be the non-empty intersection of elements from $\{\mathcal{B}^k\}_{k\in\mathbb{N}_0}$ and $\{\mathcal{B}^r\}_{k\in\mathbb{N}_0}$. Then clearly the number of $\mathcal{B}^s$ which are contained in a given $\mathcal{B}^k$ is bounded independently of k and we call this bound to be M. Therefore

$$\sum_{k\in\mathbb{N}_0} ||\psi.1_{\mathcal{B}^s}||_\infty \le \sum_{k\in\mathbb{N}_0} \sum_{\mathcal{B}^s\subset\mathcal{B}^k} ||\psi.1_{\mathcal{B}^k}||_\infty \le M \sum_{k\in\mathbb{N}_0} ||\psi.1_{\mathcal{B}^k}||_\infty,$$

$$\sum_{r\in\mathbb{N}_0} ||\psi.1_{\mathcal{B}^r}||_\infty = \sum_{s\in\mathbb{N}_0} ||\psi.1_{\mathcal{B}^{s(j)}}||_\infty \le \sum_{j\in\mathbb{N}_0} ||\psi.1_{\mathcal{B}^j}||_\infty,$$

where, $s(j)$ is such that $\mathcal{B}^{s(j)} \subset \mathcal{B}^r$ and $||\psi.1_{\mathcal{B}^r}||_\infty = ||\psi.1_{\mathcal{B}^{s(j)}}||_\infty$. This implies $||\psi||_{W,q} \le M\,||\psi||_{W,\mathbf{p}}$ and similar argument gives the opposite inequality.

(ii) Making the assumption $|\mathbf{p}| \le |q|$ implies $M = 2$.

(iii) For the part (iii), the proof follows on the similar lines as in (i).

Definition 3.3 Given a function ψ, we say $\psi \in \widehat{W}(L^\infty(\mathbb{K}), L^1(\mathbb{K}))$, if

$$||\psi||_{\widehat{W},\mathbf{p}} = \sum_{k\in\mathbb{N}_0} (||\widehat{\psi}.1_{\mathcal{B}^k}||_\infty + ||\widehat{\psi}.1_{\mathcal{B}^{-k}}||_\infty) < \infty.$$

For the functions in $\widehat{W}(L^\infty(\mathbb{K}), L^1(\mathbb{K}))$, the following properties hold.

Lemma 3.4 (i) If $||\psi||_{\widehat{W},\mathfrak{p}}$ is finite for some $\mathfrak{p}$, then it is finite for all $\mathfrak{p}$.
(ii) If $|\mathfrak{p}| \leq |q|$, then $||\psi||_{\widehat{W},q} \leq 2||\psi||_{\widehat{W},\mathfrak{p}}$.
(iii) For any $\mathfrak{p}, q$, we have

$$\| \, |q|^{\frac{1}{2}}\psi(q^r) \, \|_{\widehat{W},\mathfrak{p}} \leq 2||\psi||_{\widehat{W},\mathfrak{p}}.$$

The proof follows on the similar lines as in case of Lemma 3.2.

Lemma 3.5 Let $\psi \in L^2(\mathbb{K})$ be such that
(i) there exists $A, B > 0$ such that
$0 < A \leq \sum_{k\in\mathbb{N}_0} |\widehat{\psi}(\mathfrak{p}^{\mathbf{k}}\gamma)|^2 \leq B < \infty, for\ a.e\ \gamma \in \mathbb{K}.$
(ii) $\sum_{k\in\mathbb{N}} \beta\{q^{-1}u(k)\}^{\frac{1}{2}}\beta\{-q^{-1}u(k)\}^{\frac{1}{2}} = 0,$

where, $\beta(s) = \operatorname{esssup} \sum_{k\in\mathbb{N}_0} \widehat{\psi}(\mathfrak{p}^{\mathbf{k}}\gamma)\widehat{\psi}(\mathfrak{p}^{\mathbf{k}}\gamma - s)$, then there exists $q_0 > 0$, such that $\left\{q^{\frac{n}{2}}\psi(\mathfrak{p}^n(x - qu(m)))\right\}_{m,n\in\mathbb{N}_0}$ generates a frame for $L^2(\mathbb{K})$ for each $0 < q < q_0$.
We can re-state the above Lemma as
Let $\psi \in L^2(\mathbb{K})$ such that
(i) there exists constants $A, B > 0$, such that
$A < \sum_{n\in\mathbb{N}_0} |\psi(x - \mathfrak{p}u(n))|^2 < B.$

(ii) $\sum_{k\in\mathbb{N}} \beta\{q^{-1}u(k)\} = 0,$
where, $\beta(s) = \operatorname{esssup}| \sum_{n\in\mathbb{N}_0} \psi(x - \mathfrak{p}u(n))\overline{\psi(x - s - \mathfrak{p}u(n))}|$, then there exists $q_0 > 0$,such that $\left\{q^{\frac{n}{2}}\psi(\mathfrak{p}^n(x - qu(m)))\right\}_{m,n\in\mathbb{N}_0}$ generates a frame.
Proof. For fixed n, consider $\frac{1}{q}$-periodic function given by

$$F_n(t) = \sum_{k\in\mathbb{N}_0} \phi(t - q^{-1}u(k))\psi(t - \mathfrak{p}u(n) - q^{-1}u(k)).$$

Now, since, $F_n \in \mathcal{B}$ and both ϕ, ψ are bounded and

$$\int_{\mathbb{K}} \phi(t)\overline{\psi(t - \mathfrak{p}u(n))} \ \overline{\chi_m(qt)}dt = \int_{\mathcal{B}} F_n(t)\, \overline{\chi_m(qt)}dt.$$

Also, we have by Plancheral formula

$$\sum_{m\in\mathbb{N}_0} | \int_{\mathcal{B}} F_n(t)\, \overline{\chi_m(qt)}dt|^2 = q^{-1} \int_{\mathcal{B}} |F_n(t)|^2dt.$$

Therefore,

$$\sum_{n\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0}|\langle\phi,\psi(t-\mathfrak{p}u(n)\chi_m(qt))\rangle|^2$$

$$=\sum_{n\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0}\left|\int_{\mathbb{K}}\phi(t)\overline{\psi(t-\mathfrak{p}u(n))\chi_m(qt)}dt\right|^2$$

$$=q^{-1}\sum_{n\in\mathbb{N}_0}\int_{\mathcal{B}}\left|\sum_{k\in\mathbb{N}_0}\phi(t-q^{-1}u(k))\psi(t-\mathfrak{p}u(n)-q^{-1}u(k))\right|^2dt$$

$$=q^{-1}\sum_{n\in\mathbb{N}_0}\int_{\mathcal{B}}\sum_{r\in\mathbb{N}_0}\overline{\phi(t-q^{-1}u(r))}\psi(t-\mathfrak{p}u(n)-q^{-1})dt$$

$$\times\sum_{k\in\mathbb{N}_0}\phi(t-q^{-1}u(k))\,\overline{\psi(t-\mathfrak{p}u(n)-q^{-1}u(k))}\,dt$$

$$=q^{-1}\sum_{n\in\mathbb{N}_0}\sum_{r\in\mathbb{N}_0}\int_{\mathcal{B}}\overline{\phi(t-q^{-1}u(r))}\psi(t-\mathfrak{p}u(n)-q^{-1}u(r))dt$$

$$\times\sum_{k\in\mathbb{N}_0}\phi(t-q^{-1}u(k))\,\overline{\psi(t-\mathfrak{p}u(n)-q^{-1}u(k))}dt$$

$$=q^{-1}\sum_{n\in\mathbb{N}_0}\int_{\mathbb{K}}\overline{\phi(t)}\psi(t-\mathfrak{p}u(n))\ \sum_{k\in\mathbb{N}_0}\phi(t-q^{-1}u(k))\overline{\psi(t-\mathfrak{p}u(n)-q^{-1}u(k))}\,dt$$

$$=q^{-1}\sum_{k\in\mathbb{N}_0}\int_{\mathbb{K}}\overline{\phi(t)}\,\phi(t-q^{-1}u(k))\times\sum_{n\in\mathbb{N}_0}\psi(t-\mathfrak{p}u(n))\,\overline{\psi(t-\mathfrak{p}u(n)-q^{-1}u(k))}dt$$

$$=q^{-1}\int_{\mathbb{K}}|\phi(t)|\ \sum_{n\in\mathbb{N}_0}|\psi(t-\mathfrak{p}u(n))|^2\,dt+q^{-1}\sum_{k\in\mathbb{N}}\int_{\mathbb{K}}\overline{\phi(t)}\,\phi(t-q^{-1}u(k))$$

$$\times\sum_{n\in\mathbb{N}_0}\psi(t-\mathfrak{p}u(n))\,\overline{\psi(t-\mathfrak{p}u(n)-q^{-1}u(k))}dt=\Delta.$$

By Cauchy-Schwarz inequality, we have

$$\Delta\le\ q^{-1}B||\phi||_2^2\ +\ q^{-1}\sum_{k\in\mathbb{N}}\beta(q^{-1}u(k))\int_{\mathbb{K}}\overline{\phi(t)}\,\phi(t-q^{-1}u(k))dt\le\ B_0(q)\ ||\phi||_2^2,$$

and

$$\Delta\ge q^{-1}A||\phi||^2\ -q^{-1}\sum_{k\in\mathbb{K}}\beta(q^{-1}u(k))\int_{\mathbb{K}}\overline{\phi(t)}\,\phi(t-q^{-1}u(k))dt\ \ge\ A_0(q)\ ||\phi||_2^2,$$

where

$$A_0(q)\ =\ q^{-1}A\ -\ b^{-1}\sum_{k\in\mathbb{N}}\beta(q^{-1}u(k)),$$

and

$$B_0(q)\ =\ q^{-1}B\ +\ q^{-1}\sum_{k\in\mathbb{N}}\beta(q^{-1}u(k)).$$

By second condition, we have $q_0>\ 0$, so that $A_0(q)\ >\ 0$ and $B_0(q)\ <\ \infty$ for all $0\ <q\ <q_0$ and since $\phi\in L^2(\mathbb{K})$, we can find a sequence of compactly supported functions ϕ_j such that $\phi_j\ \rightarrow\ \phi$ in $L^2(\mathbb{K})$ as $j\rightarrow\infty$. By the above result, we have

$$A_0(q)\ ||\phi_j||_2^2\le\sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0}|\langle\phi,\chi_m(qt)\psi(t-\mathbf{p}u(n))\rangle|^2\le B_0(q)\ ||\phi_j||_2^2,$$

and thus the result follows.

The next theorem gives a necessary condition for the system $\left\{q^{\frac{n}{2}}\psi(t-u(m))\right\}_{m,n\in\mathbb{N}_0}$ to generate a frame for $L^2(\mathbb{K})$.

Theorem 3.6 For $q > 0$, if $\left\{q^{\frac{n}{2}}\psi(t-qu(m))\right\}_{m,n\in\mathbb{N}_0}$ generates a frame for $L^2(\mathbb{K})$, then

$$G(\gamma) = \sum_{n\in\mathbb{N}_0} |\psi(\mathbf{p}^n\gamma)|^2 \leq B < \infty, a.e\ \gamma \in \mathbb{K}.$$

Proof. Assume that $\operatorname{esssup} G(\gamma) = +\infty$. Given $M > 0$, we can find $\mathcal{B} \subset \mathcal{B}'$, such that $G(\gamma) > q\,M$. Let $\widehat{\phi} = \chi_{\mathcal{B}'}$, then

$$\sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0} |\left\langle \phi, \mathbf{p}^{\frac{-n}{2}}\psi\left(\mathbf{p}^n(t-qu(m))\right)\right\rangle|^2 =$$
$$\sum_{n\in\mathbb{N}_0}\mathbf{p}^{-n}\sum_{m\in\mathbb{N}_0} |\int_{\mathcal{B}} \overline{\widehat{\psi}(\mathbf{p}^n\gamma)}\chi_m(q\mathbf{p}^{-n}\gamma)\,d\gamma|^2.$$

Since, $\left\{q^{\frac{n}{2}}\chi_m(q^n\gamma)\right\}_{m\in\mathbb{N}_0}$ is an orthonormal basis for $L^2(\mathfrak{p}^n\mathcal{B})$

and $\overline{\widehat{\psi}(q^n\gamma)}.\chi_B \in L^2(\mathcal{B}) \subseteq L^2(\mathfrak{p}^n\mathcal{B})$, therefore

$$\sum_{m\in\mathbb{N}_0} q^n\sum_{n\in\mathbb{N}_0} |\int_{\mathbb{B}} \overline{\widehat{\psi}(\mathfrak{p}^n\gamma)}\chi_m(q^n\gamma)d\gamma|^2 = \sum_{n\in\mathbb{N}_0} q\int_{\mathbb{B}} |\overline{\widehat{\psi}(q^n\gamma)}|^2 d\gamma > M||\phi||^2$$

and since M is arbitrary, therefore $\left\{q^{\frac{n}{2}}\psi(\mathfrak{p}^n(x-u(m)))\right\}_{m,n\in\mathbb{N}_0}$ cannot generate a frame for $L^2(\mathbb{K})$.

The next theorem gives a necessary condition for $\left\{q^{\frac{n}{2}}\psi(\mathfrak{p}^n(x-u(m)))\right\}_{m,n\in\mathbb{N}_0}$ to generate a frame for $L^2(\mathbb{K})$ irrespective of the fact whether ψ has compact support or not.

Theorem 3.7 If $\left\{q^{\frac{n}{2}}\psi(\mathfrak{p}^n(x-u(m)))\right\}_{m,n\in\mathbb{N}_0}$ is a frame for $L^2(\mathbb{K})$, then

$$0 < A \leq G(\gamma) = \sum_{n\in\mathbb{N}_0} |\widehat{\psi}(\mathfrak{p}^n\gamma)|^2 \leq B < +\infty.$$

Proof. Assume that $\operatorname{essinf} G(\gamma) = 0$, given $\delta > 0$, we can find $\mathcal{B} \subset \mathcal{B}'$ and $G(\gamma) < \delta$ on $\mathcal{B}$. If we set $\widehat{\phi} = \chi_{\mathcal{B}}$

$$\sum_{n\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0} |\left\langle \phi, q^{\frac{n}{2}}\psi(\mathfrak{p}^n(x-u(m)))\right\rangle|^2 = \sum_{n\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0} q^n\,|\langle\widehat{\phi}, \overline{\chi_m(q^n\gamma)}\widehat{\psi}(q^n\gamma)\rangle|^2$$

$$= \sum_{n\in\mathbb{N}_0} q^n\sum_{m\in\mathbb{N}_0} |\int_{\mathcal{B}} \overline{\widehat{\psi}(q^n)}\chi_m(q^n\gamma)\,d\gamma|^2.$$

For fixed $n \in \mathbb{N}_0$, $\left\{q^{\frac{n}{2}}\chi_m(q^n\gamma)\right\}_{m,n\in\mathbb{N}_0}$ is an orthonormal basis for $L^2(\mathfrak{p}^n\mathcal{B}')$ and $\overline{\widehat{\psi}(\mathbf{p}^{-n}\gamma)}.\mathcal{B} \in L^2(\mathbf{p}^n\mathcal{B}')$, then

$$\sum_{n\in\mathbb{N}_0}\mathbf{p}^{-n}\sum_{m\in\mathbb{N}_0} |\int_{\mathcal{B}} \overline{\widehat{\psi}(q^n\gamma)}\chi_m(q^n\gamma)\,d\gamma|^2 = \sum_{n\in\mathbb{N}_0}\int_{\mathcal{B}} |\overline{\widehat{\psi}(q^n\gamma)}|^2\,d\gamma < \delta\,||\phi||^2.$$

Since δ is arbitrary, therefore $\left\{q^{\frac{n}{2}}\psi(\mathfrak{p}^n(x-u(m)))\right\}_{m,n\in\mathbb{N}_0}$ is not a frame for $L^2(\mathbb{K})$.

The next theorem gives a sufficient condition for $\left\{q^{\frac{n}{2}}\psi(\mathfrak{p}^n(x-u(m)))\right\}_{m,n\in\mathbb{N}_0}$ to generate a frame for $L^2(\mathbb{K})$.

Theorem 3.8 Given $\psi \in \widehat{W}(L^{\infty}(\mathbb{K}), L^{1}(\mathbb{K}))$ such that $\widehat{\psi} \in W(L^{\infty}(\mathbb{K}), L^{1}(\mathbb{K}))$, if there exists constants A and B such that

$$0 < A \le \sum_{n\in\mathbb{N}_0} |\widehat{\psi}(\mathfrak{p}^n\gamma)|^2 \le B < \infty \; a.e \; for \; some \; \mathfrak{p},$$

then $\left\{q^{\frac{n}{2}}\psi(\mathfrak{p}^n(x-u(m)))\right\}_{m,n\in\mathbb{N}_0}$ generates a frame for $L^2(\mathbb{K})$ for all $\mathfrak{p}$.

Proof. By lemma (3.5), we need only to show that

$$\sum_{k\in\mathbb{N}} \beta(u(k))\beta(-u(k))^{\frac{1}{2}} = 0.$$

For fixed $\mathfrak{p}$, we claim that for all functions ϕ, f, we always have,

$$\sum_{k\in\mathbb{K}} \| \sum_{n\in\mathbb{N}_0} |\widehat{\phi}(\mathfrak{p}^n\gamma)|\,|\hat{f}(\mathfrak{p}^n\gamma - u(k))| \,\|_\infty \le 4\,||\widehat{\phi}||_{\widehat{W},\mathfrak{p}}\,||f||.$$

Consider,

$$\sum_{k\in\mathbb{N}} \| \sum_{n\in\mathbb{N}_0} \left|\widehat{\phi}(\mathfrak{p}^n\gamma)\right|\left|\hat{f}\left(\mathfrak{p}^n\gamma - u(k)\right)\right| \|_\infty$$

$$\le \sum_{k\in\mathbb{N}_0} \| \sum_{n\in\mathbb{N}_0} |\widehat{\phi}(\mathfrak{p}^n\gamma)||\hat{f}(\mathfrak{p}^n\gamma - u(k))|\chi_{\mathcal{B}} \|_\infty$$

$$\le \sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}_0} \| \widehat{\phi}(\mathfrak{p}^n\gamma).\chi_{\mathcal{B}} \|_\infty \| \hat{f}(\mathfrak{p}^n\gamma - u(k)) \|_\infty$$

(By Cauchy-Schwarz inequality)

$$= \sum_{n\in\mathbb{N}_0} \| \widehat{\phi}(\mathfrak{p}^n\gamma).\chi_{\mathcal{B}} \|_\infty \sum_{k\in\mathbb{N}} \| \widehat{\phi}(\mathfrak{p}^n\gamma - u(k))\chi_{\mathcal{B}} \|_\infty$$

$$\le \sum_{n\in\mathbb{N}_0} \| \widehat{\phi}(\gamma)\chi_{\mathcal{B}} \|_\infty \sum_{k\in\mathbb{N}} \| \widehat{\phi}(\mathfrak{p}^n\gamma - u(k))\chi_{\mathcal{B}} \|_\infty$$

$$\le \sum_{n\in\mathbb{N}_0} \| \widehat{\phi}(\gamma)\chi_{\mathcal{B}} \|_\infty . 2 \| \hat{f}(\mathfrak{p}^n\gamma) \|_{W,\mathfrak{p}}$$

$$= \sum_{n\in\mathbb{N}_0} \| \widehat{\phi}(\gamma)\chi_{\mathcal{B}} \|_\infty . 2 \| \hat{f}(\gamma) \|_{W,\mathfrak{p}^n}$$

$$\le 4 \sum_{n\in\mathbb{N}_0} \| \widehat{\phi}\chi_{\mathcal{B}} \|_\infty . \| \hat{f} \|_{W,\mathfrak{p}}$$

$$= 4 \| \widehat{\phi} \|_{W,\mathfrak{p}} \| \hat{f} \|_{W,\mathfrak{p}}.$$

Now we fix $\varepsilon > 0$, let N be so large that $\sum_{n\ge\mathbb{N}} (\| \widehat{\psi}.\chi_{\mathcal{B}^n} \|_\infty + \| \widehat{\psi}.\chi_{\mathcal{B}^{-n}} \|_\infty) < \varepsilon$ and $\sum_{n\ge\mathbb{N}} \| \widehat{\phi}.\chi_{\mathcal{B}^n} \|_\infty < \varepsilon$. Let $\psi_0 = \widehat{\psi}.1_{\mathcal{B}^n}$ and $\psi_1 = \widehat{\psi} - \widehat{\psi}_0$, then $\| \psi_1 \|_{\widehat{W},\mathfrak{p}} < \varepsilon$, $\| \psi_1 \|_{W,\mathfrak{p}} < \varepsilon$ and

$$\sum_{k\in\mathbb{N}} \beta(u(k)) = \sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}_0} \| \widehat{\psi}(\mathfrak{p}^n\gamma)\widehat{\psi}(\mathfrak{p}^n\gamma - u(k)) \|_\infty$$

$\leq \sum_{k\in\mathbb{N}} \| \sum_{n\in\mathbb{N}_0} |\hat{\psi}(\mathfrak{p}^n\gamma)|\hat{\psi}(\mathfrak{p}^n\gamma - u(k))| \|_\infty$

$= \sum_{k\in\mathbb{N}} \| \sum_{n\in\mathbb{N}_0} |\psi_0(\mathfrak{p}^n\gamma) + \psi_1(\mathfrak{p}^n\gamma)||\psi_0(\mathfrak{p}^n\gamma - u(k)) + \psi_1(\mathfrak{p}^n\gamma - u(k))| \|_\infty$

$\leq \sum_{k\in\mathbb{N}} \| \sum_{n\in\mathbb{N}_0} |\psi_0(\mathfrak{p}^n\gamma)||\psi_0(\mathfrak{p}^n\gamma - u(k))| \|_\infty + \sum_{k\in\mathbb{N}} \| \sum_{n\in\mathbb{N}_0} |\psi_0(\mathfrak{p}^n\gamma)||\psi_1(\mathfrak{p}^n\gamma - u(k))| \|_\infty$

$+ \sum_{k\in\mathbb{N}} \| \sum_{n\in\mathbb{N}_0} |\psi_1(\mathfrak{p}^n\gamma)||\psi_0(\mathfrak{p}^n\gamma - u(k))| \|_\infty + \sum_{k\in\mathbb{N}} \| \sum_{n\in\mathbb{N}_0} |\psi_1(\mathfrak{p}^n\gamma)||\psi_1(\mathfrak{p}^n\gamma - u(k))| \|_\infty$

$= \sum_{k\in\mathbb{N}} \| \sum_{n\in\mathbb{N}_0} |\psi_0(\mathfrak{p}^n\gamma)||\psi_1(\mathfrak{p}^n\gamma - u(k))| \|_\infty + \sum_{k\in\mathbb{N}} \| \sum_{n\in\mathbb{N}_0} |\psi_1(\mathfrak{p}^n\gamma)||\psi_0(\mathfrak{p}^n\gamma - u(k))| \|_\infty$

$+ \sum_{k\in\mathbb{N}} \| \sum_{n\in\mathbb{N}_0} |\psi_1(\mathfrak{p}^n\gamma)| |\psi_1(\mathfrak{p}^n\gamma - u(k))| \|_\infty$

$\leq 4|\psi_0||_{\widehat{W},\mathfrak{p}}. \| \psi_1 \|_{W,\mathfrak{p}} + 4 \| \psi_1 \|_{\widehat{W},\mathfrak{p}}. \| \psi_0 \|_{W,\mathfrak{p}} + 4 \| \psi_1 \|_{\widehat{W},\mathfrak{p}}. \| \psi_1 \|_{W,\mathfrak{p}}$

$< 4\varepsilon \| \hat{\psi} \|_{\widehat{W},\mathfrak{p}} + 4\varepsilon^2.$

Similarly, we have

$$\beta(-u(k)) < 4\varepsilon \| \hat{\psi} \|_{\widehat{W},\mathfrak{p}} + 4\varepsilon \| \hat{\psi} \|_{W,\mathfrak{p}} + 4\varepsilon^2$$

$\sum_{k\in\mathbb{N}} \beta(u(k))^{\frac{1}{2}}\beta(-u(k))^{\frac{1}{2}} \leq \{\sum_{k\in\mathbb{N}} \beta(u(k))\}^{\frac{1}{2}}.\{\sum_{k\in\mathbb{N}} \sum_{k\in\mathbb{N}} \beta(-u(k))\}^{\frac{1}{2}} < 4\varepsilon \| \hat{\psi} \|_{\widehat{W},\mathfrak{p}} + 4\varepsilon^2,$

and thus the result follows.

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Synchronization of 3D Autonomous Chaotic System using Active Nonlinear Control

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Abstract
This manuscript presents the complete synchronization of identical 3D autonomous chaotic systems. Active nonlinear control is used to compute the controllers to achieve the complete synchronization of 3D autonomous chaotic system. Numerical simulations are provided to illustrate the effectiveness of the proposed synchronization schemes for identical 3D autonomous chaotic systems.
Keywords: 3D autonomous chaotic system, complete synchronization, nonlinear active control.

1. Introduction

Much has been written and said about the concept of synchronization of chaotic systems since it was first introduced by Pecora and Caroll [1]. Because of its interdisciplinary nature the chaos synchronization problem has received interest from researchers across the academic fields such as physics, mathematics, engineering, biology, chemistry, etc. The potential applications of chaos synchronization to engineering systems, information processing, secure communications, and biomedical science amongst many others has led to a vast variety of research studies in this topic of nonlinear science [2, 3, 4, 5].

The control of chaos is concerned with using some designed control input(s) to modify the characteristics of a parameterized nonlinear system. A number of methods such as active control [6], adaptive control [7], backstepping control [8] and optimal control [9], sliding mode control [10], adaptive sliding mode control [11, 12] exist for the control of chaos in systems.

Various kinds of synchronization such as sequential, phase, anticipated, measure, generalized, lag projective synchronization [13], complete synchronization [14], hybrid synchronization [15], anti-synchronization [16], projective synchronization [17] and hybrid function projective synchronization [7] have been developed and are frequently used.

In recent years, active control [6, 17] have been widely recognized as powerful design methods to control and synchronize chaos. Active control technique gives the flexibility to construct a control law so that it can be used widely to control and synchronize various nonlinear systems, including chaotic systems.

This manuscript is categorize as follows: In section 2 system description of 3D autonomous chaotic system are given. In section 3 complete synchronization of 3D autonomous chaotic system is achieved. In section 4 simulation results are discussed. Finally in section 5 concluding remarks are given.

2. System Description of 3D Autonomous Chaotic System

The dynamics of 3D autonomous chaotic system [13] having seven terms with three quadratic terms is given by

$$\begin{cases} \dot{x}_1 &= x_2 - ax_1 + 10x_2x_3 \\ \dot{x}_2 &= cx_2 + 5x_1x_3 \\ \dot{x}_3 &= bx_3 - 5x_1x_2 \end{cases} \tag{1}$$

where x_1, x_2, x_3 are the variables and a, b, c are the parameters. When the parameters a=0.4, b=0.175, c=-0.4 and initial condition (0.349,0.113,0.2) are chosen then the system displays chaotic attractor (two strange attractors) as shown in figure (1). The corresponding Lyapunov exponents as shown in figure (2) of the 3D autonomous chaotic attractor are $\gamma_1 = 0.071025, \gamma_2 = -0.000032932, \gamma_3 = -0.69599$.

The Kaplyan-Yorke dimension is defined by

$$\begin{aligned} D &= j + \sum_{i=1}^{j} \frac{\gamma_i}{|\gamma_{j+1}|} \\ &= 2.10209 \end{aligned}$$

where j is the largest integer satisfying $\sum_{i=1}^{j} \gamma_j \geq 0$ and $\sum_{i=1}^{j+1} \gamma_j < 0$. Therefore Kaplan-Yorke dimension of the chaotic attractor is $D = 2.10209$ which means that the Lyapunov dimension of the chaotic attractor is fractional.

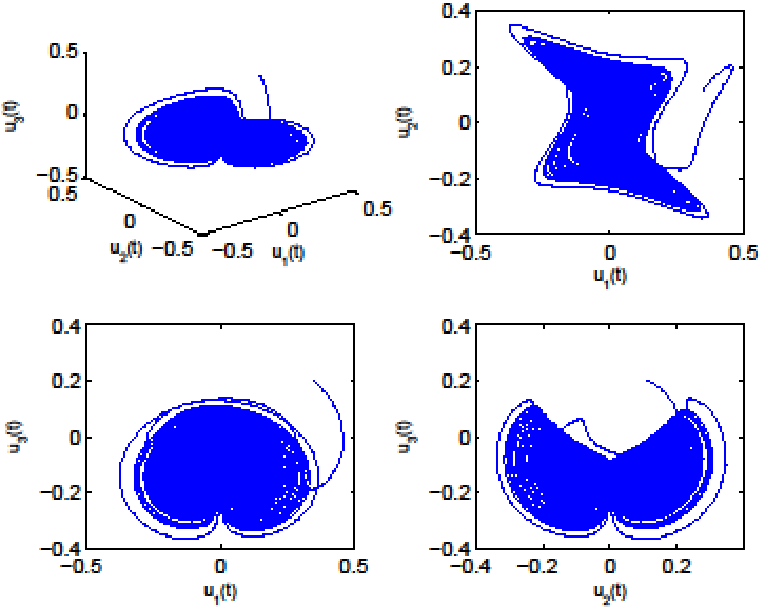


Figure 1: Phase Portrait of the 3D autonomous chaotic system

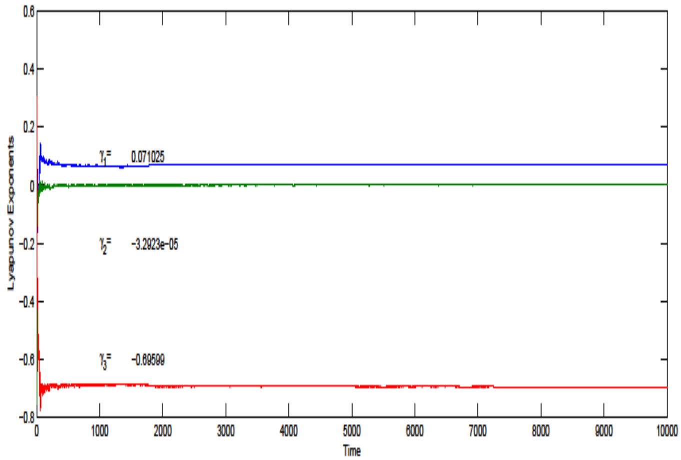


Figure 2: Lyapunov exponents of the novel chaotic system

2.1 Poincaré map, Bifurcation diagram and Impact of system parameters

As an important analysis technique, the Poincaré map reflects the periodic and chaotic behavior of the system. When $a = 0.4, b = 0.175, c = -0.4$ one may take $x_1 = 0$ as the crossing section as shown in Figure 3.

The bifurcation diagram of $|x_1|$ with respect to parameters a, b, c is shown in Figure 4 which shows abundant and complex dynamical behaviors.

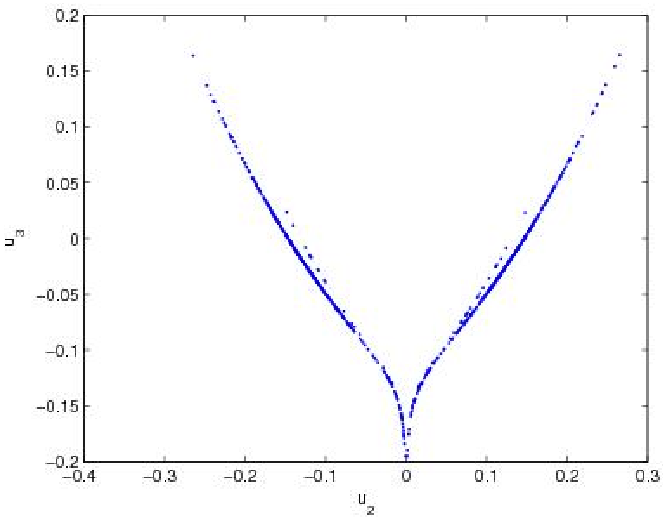


Figure 3: Poincare map on the crossing section $x_1 = 0$

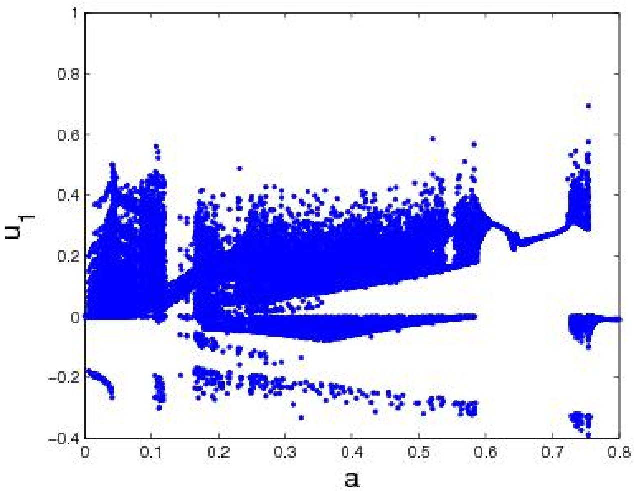


Figure 4: Bifurcation diagram of $|x_1|$ versus a

3. Complete Synchronization of 3D Autonomous Chaotic System

We consider the identical 3D autonomous chaotic system to achieve the complete synchronization between the drive system (2) and the response system (3) respectively.

$$\begin{cases} \dot{x}_1 &= x_2 - ax_1 + 10x_2x_3 \\ \dot{x}_2 &= cx_2 + 5x_1x_3 \\ \dot{x}_3 &= bx_3 - 5x_1x_2 \end{cases} \tag{2}$$

$$\begin{cases} \dot{y}_1 &= y_2 - ay_1 + 10y_2y_3 + u_1 \\ \dot{y}_2 &= cy_2 + 5y_1y_3 + u_2 \\ \dot{y}_3 &= by_3 - 5y_1y_2 + u_3 \end{cases} \tag{3}$$

where $x_1, x_2, x_3, y_1, y_2, y_3$ are the state vectors and a, b, c are the parameters of the system and $u = (u_1 \quad u_2 \quad u_3)^t$ is the nonlinear controller to be designed.

The error states for complete synchronization are defined as

$$\begin{cases} e_1 = y_1 - x_1 \\ e_2 = y_2 - x_2 \\ e_3 = y_3 - x_3 \end{cases} \tag{4}$$

3.1 Design of Control Function

The error dynamics is obtained as

$$\begin{cases} \dot{e}_1 &= e_2 - ae_1 + 10y_2y_3 - 10x_2x_3 + u_1 \\ \dot{e}_2 &= ce_2 + 5y_1y_3 - 5x_1x_3 + u_2 \\ \dot{e}_3 &= be_3 - 5y_1y_2 + 5x_1x_2 + u_3 \end{cases} \tag{5}$$

To achieve asymptotic stability of system (5), we eliminate terms which cannot be expressed as linear terms in e_1, e_2, e_3 as follows :

$$\begin{cases} u_1 &= -10y_2y_3 + 10x_2x_3 + v_1 \\ u_2 &= -5y_1y_3 + 5x_1x_3 + v_2 \\ u_3 &= 5y_1y_2 - 5x_1x_2 + v_3 \end{cases} \tag{6}$$

Substituting (6) into (5), we obtain

$$\begin{cases} \dot{e}_1 &= e_2 - ae_1 + v_1 \\ \dot{e}_2 &= ce_2 + v_2 \\ \dot{e}_3 &= be_3 + v_3 \end{cases} \tag{7}$$

Using the active control method, a constant matrix A is chosen such that the error dynamics (7) is controlled. For that the feedback matrix is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \tag{8}$$

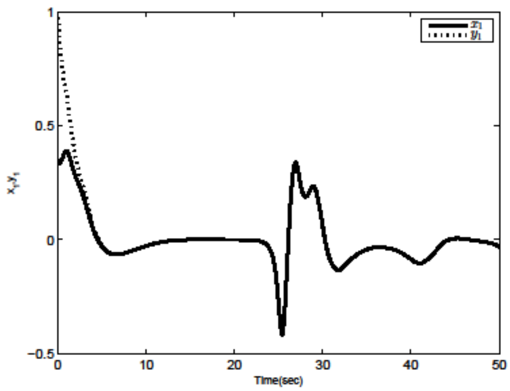
with

$$A = \begin{bmatrix} \lambda_1 + a & -1 & 0 \\ 0 & \lambda_2 - c & 0 \\ 0 & 0 & \lambda_3 - b \end{bmatrix} \tag{9}$$

In (9) the six eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are chosen to be negative in order to achieve a stable complete synchronization between two identical 3D autonomous chaotic system.

3.2 Numerical Simulation

Numerical solutions are carried out in Matlab to solve systems (2) and (3) with the following initial conditions $(x_1, x_2, x_3) = (0.349,0,0.16)$ and $(y_1, y_2, y_3) = (1,0.113,0.2)$. The system parameters are chosen as $a = 0.4$, $b = 0.175$, $c = -0.4$, so the system behaves chaotically as shown in figure (1). Figure (5) shows the dynamics of the state variables (x and y) of the master system and the slave system. The error dynamics of the system when the controls are activated at time $t = 0$ is shown in figure (6). The synchronization errors between the two system is seen to converge to zero.



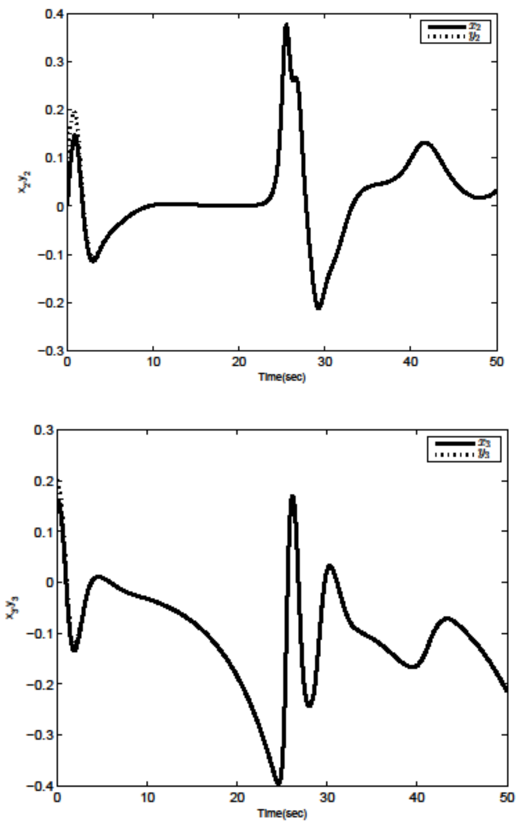


Figure 5: Dynamics of the state variables when the control functions are activated at $t = 0$

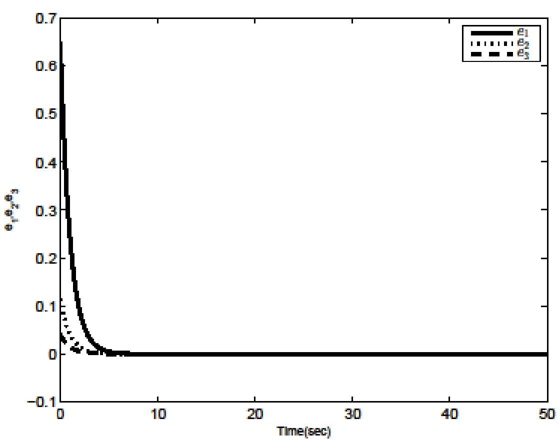


Figure 6: Error dynamics of the state variables when the control functions are activated for $t = 0$

4 Conclusion

This manuscript demonstrates that the chaos synchronization of 6D hyperchaotic systems using active control method is achieved. Numerical simulations are used to verify the effectiveness of the proposed nonlinear active control technique. Computational and analytical results are in excellent agreement.

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A q-Dunkl generalization of modified Szász-operators

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Abstract
The main purpose of this article is to introduce a modification of q-Dunkl generalization of Szász-operators. We obtain approximation results via well known Korovkin's type theorem . we obtain Further, we obtain the order of approximation, rate of convergence, functions belonging to the Lipschitz class and some direct theorems .
***Keywords and phrases*:** q-integers; Dunkl analogue; Szász operator; q- Szász-Mirakjan-Kantorovich; modulus of continuity; Peetre's K-functional.
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1. Introduction and preliminaries

In 1950, for $x \geq 0$, Szász [19] introduced the operators

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), f \in C[0,\infty). \tag{1.1}$$

In the field of approximation theory, the application of q-calculus emerged as a new area in the field of approximation theory. The first q-analogue of the well-known Bernstein polynomials was introduced by Lupaş by applying the idea of q-integers [5]. In 1997, Phillips [17] considered another q-analogue of the classical Bernstein polynomials. Later on, many authors introduced q-generalizations of various operators and investigated several approximation properties [7, 8, 9, 10, 11, 13, 14, 15, 16, 20, 21].

We now present some basic definitions and notations of the q-calculus which are used in this paper.

Definition 1.1 *For $|q| < 1$, the q-number $[\lambda]_q$ is defined by*

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases} \tag{1.2}$$

Definition 1.2 *For $|q| < 1$, the q-factorial $[n]_q!$ is defined by*

$$[n]_q! = \begin{cases} 1 & (n = 0) \\ \prod_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases} \tag{1.3}$$

Sucu [18] defined a Dunkl analogue of Szász operators via a generalization of the exponential function as follows:

$$S_n^*(f;x) := \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} f\left(\frac{k+2\mu\theta_k}{n}\right), \tag{1.4}$$

where $x \geq 0, f \in C[0,\infty), \mu \geq 0, n \in \mathbb{N}$.
Cheikh et al., [2] stated the q-Dunkl classical q-Hermite type polynomials and gave definitions of q-Dunkl analogues of exponential functions and recursion relations for $\mu > -\frac{1}{2}$ and $0 < q < 1$.

$$e_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_{\mu,q}(n)}, x \in 0,\infty) \tag{1.5}$$

$$E_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{\gamma_{\mu,q}(n)}, x \in 0,\infty) \tag{1.6}$$

where

$$\gamma_{\mu,q}(n) = \frac{(q^{2\mu+1},q^2)_{\left[\frac{n+1}{2}\right]}(q^2,q^2)_{\left[\frac{n}{2}\right]}}{(1-q)^n}\gamma_{\mu,q}(n), n \in \mathbb{N}. \tag{1.7}$$

In [4], Içöz gave the Dunkl generalization of Szász operators via q-calculus as:

$$D_{n,q}(f;x) = \frac{1}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} f\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right), \tag{1.8}$$

for $\mu > \frac{1}{2}, x \geq 0, 0 < q < 1$ and $f \in C[0,\infty)$.
Previous studies demonstrate that providing a better error estimation for positive linear operators plays an important role in approximation theory, which allows us to approximate much faster to the function being approximated.
Motivated essentially by Içöz [4] the recent investigation of Dunkl generalization of Szász-Mirakjan operators via q-calculus we show that our modified operators have better error estimation than [4]. We also prove several approximation results and successfully extend the results of [4]. Several other related results have also been discussed.

2. Construction of operators and moments estimation

Let $\{a_n\}$ and $\{b_n\}$; are increasing and unbounded sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{1}{b_n} \to 0, and\, \frac{a_n}{b_n} = 1 + O\left(\frac{1}{b_n}\right). \tag{2.1}$$

For any $\frac{1}{2n} \leq x < \frac{1}{1-q^n}, 0 < q < 1, \mu > \frac{1}{2n}$ and $n \in \mathbb{N}$ we define

$$D_{n,q}^{a_n,b_n}(f;x) = \frac{1}{e_{\mu,q}([n]_q x a_n)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^k}{\gamma_{\mu,q}(k)} f\left(\frac{1-q^{2\mu\theta_k+k}}{b_n(1-q^n)}\right), \tag{2.2}$$

where $e_{\mu,q}(x), \gamma_{\mu,q}$ are defined in (1.5), (1.7) by [18] and $f \in C_\zeta[0,\infty)$ with $\zeta \geq 0$ and

$$C_\zeta[0,\infty) = \{f \in C[0,\infty): |f(t)| \leq M(1+t)^\zeta, for some M > 0, \zeta > 0\}. \tag{2.3}$$

Note that the parameters a_n and b_n have an important effect for a better approach of the operator $D_{n,q}^{a_n,b_n}$.

Lemma 2.1 *Let $D_{n,q}^{a_n,b_n}(.;.)$ be the operators given by (2.2). Then for each $\frac{1}{2n} \leq x < \frac{1}{b_n(1-q^n)}, n \in \mathbb{N}$, we have the following identities/ estimates:*

1. $D_{n,q}^{a_n,b_n}(1;x) = 1$,
2. $D_{n,q}^{a_n,b_n}(t;x) = \left(\frac{a_n}{b_n}\right)x$,
3. $\left(\frac{a_n}{b_n}\right)^2 x^2 + \left(\frac{a_n}{b_n}\right) q^{2\mu}[1-2\mu]_q \frac{e_{\mu,q}(\frac{a_n}{b_n}q[n]_q x)}{e_{\mu,q}([n]_q x)} \frac{x}{\frac{a_n}{b_n}[n]_q} \leq D_{n,q}^{a_n,b_n}(t^2;x)$
$$\leq \left(\frac{a_n}{b_n}\right)^2 x^2 + \left(\frac{a_n}{b_n}\right)[1+2\mu]_q \frac{x}{[n]_q}.$$

Proof. As $D_{n,q}^{a_n,b_n}(1;x) = \frac{1}{e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^k}{\gamma_\mu(k)} = 1$, and

$$D_{n,q}^{a_n,b_n}(t;x) = \frac{1}{e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^k}{\gamma_\mu(k)} \left(\frac{1-q^{2\mu\theta_k+k}}{b_n(1-q^n)}\right)$$
$$= \frac{1}{b_n[n]_q e_{\mu,q}(a_n[n]_q x)} \sum_{k=1}^{\infty} \frac{(a_n[n]_q x)^k}{\gamma_\mu(k-1)}$$
$$= \left(\frac{a_n}{b_n}\right) x$$

then (1) and (2) hold. Similarly

$$D_{n,q}^{a_n,b_n}(t^2;x) = \frac{1}{e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^k}{\gamma_\mu(k)} \left(\frac{1-q^{2\mu\theta_k+k}}{b_n(1-q^n)}\right)^2$$
$$= \frac{1}{b_n^2[n]_q^2 e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^k}{\gamma_\mu(k-1)} \left(\frac{1-q^{2\mu\theta_k+k}}{1-q}\right)$$
$$= \frac{1}{b_n^2[n]_q^2 e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^{k+1}}{\gamma_\mu(k)} \left(\frac{1-q^{2\mu\theta_{k+1}+k+1}}{1-q}\right).$$

From [4] we know that

$$[2\mu\theta_{k+1}+k+1]_q = [2\mu\theta_k+k]_q + q^{2\mu\theta_k+k}[2\mu(-1)^k+1]_q. \tag{2.4}$$

Now by separating to the even and odd terms and using (2.4), we get

$$D_{n,q}^{a_n,b_n}(t^2;x) = \frac{1}{b_n^2[n]_q^2 e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^{k+1}}{\gamma_\mu(k)} \left(\frac{1-q^{2\mu\theta_{k+1}+k+1}}{1-q}\right)$$
$$+ \frac{[1+2\mu]_q}{b_n^2[n]_q^2 e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^{2k+1}}{\gamma_\mu(2k)} q^{2\mu\theta_{2k}+2k}$$
$$+ \frac{[1-2\mu]_q}{b_n^2[n]_q^2 e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^{2k+2}}{\gamma_\mu(2k)} q^{2\mu\theta_{2k+1}+2k+1}.$$

Since

$$[1-2\mu]_q \le [1+2\mu]_q, \tag{2.5}$$

we have

$$D_{n,q}^{a_n,b_n}(t^2;x) \ge (x\frac{a_n}{b_n})^2 + \frac{x a_n[1-2\mu]_q}{b_n[n]_q e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(q a_n[n]_q x)^{2k}}{\gamma_\mu(2k)}$$
$$+ \frac{q^{2\mu} x a_n[1-2\mu]_q}{b_n[n]_q e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(q a_n[n]_q x)^{2k+1}}{\gamma_\mu(2k+1)}$$
$$\ge (x\frac{a_n}{b_n})^2 + q^{2\mu}[1-2\mu]_q \frac{e_{\mu,q}(q\frac{a_n}{b_n}[n]_q x)}{e_{\mu,q}(\frac{a_n}{b_n}[n]_q x)} \frac{x a_n}{b_n[n]_q}.$$

On the other hand, we have

$$D_{n,q}^{a_n,b_n}(t^2;x) \le (x\frac{a_n}{b_n})^2 + [1+2\mu]_q \frac{x a_n}{b_n[n]_q}.$$

This completes the proof.

Lemma 2.2 *Let the operators $D_{n,q}^{a_n,b_n}(.;.)$ be given by (2.2). Then for each $x \ge \frac{1}{2n}, n \in \mathbb{N}$, we have*

1. $D_{n,q}^{a_n,b_n}(t-x;x) = \left(\frac{a_n}{b_n} - 1\right) x$,
2. $D_{n,q}^{a_n,b_n}((t-x)^2;x) \le \left(\frac{a_n}{b_n} - 1\right)^2 x^2 + \left(\frac{a_n}{b_n}\right)[1+2\mu]_q \frac{x}{[n]_q}$.

Proof. For proof of this lemma we use

$$D_{n,q}^{a_n,b_n}(t-x;x) = D_{n,q}^{a_n,b_n}(t;x) - D_{n,q}^{a_n,b_n}(1;x),$$

And

$$D_{n,q}^{a_n,b_n}((t-x)^2;x) = D_{n,q}^{a_n,b_n}(t^2;x) - 2xD_{n,q}^{a_n,b_n}(t;x) + x^2D_{n,q}^{a_n,b_n}(1;x)$$

This ends the proof of (2).

3. Main results

We obtain the Korovkin's type approximation properties for our operators (see [1], [6]).
Let $C_B(\mathbb{R}^+)$ be the set of all bounded and continuous functions on $\mathbb{R}^+ = [0,\infty)$, which is linear normed space with

$$\| f \|_{C_B} = \sup_{x\geq 0}|f(x)|.$$

Let

$$H: = \{f: x \in 0,\infty), \frac{f(x)}{1+x^2} is\ convergent\ as\ \ x \to \infty\}.$$

Theorem 3.1 *Let $D_{n,q}^{a_n,b_n}(.;.)$ be the operators defined by (2.2). Then for any function $f \in C_\zeta[0,\infty) \cap H, \zeta \geq 2$,*

$$\lim_{n\to\infty} D_{n,q}^{a_n,b_n}(f;x) = f(x)$$

is uniformly on each compact subset of $[0,\infty)$, where $x[0,\infty)$.

Proof. The proof is based on Lemma 2.1 and well known Korovkin's theorem regarding the convergence of a sequence of linear positive operators, so it is enough to prove the conditions

$$\lim_{n\to\infty} D_{n,q}^{a_n,b_n}((t^j;x) = x^j, j = 0,1,2, \{asn \to \infty\}$$

uniformly on [0,1].

Clearly $\frac{1}{[n]_q} \to 0(n \to \infty)$ we have

$$\lim_{n\to\infty} D_{n,q}^{a_n,b_n}(t;x) = x, \lim_{n\to\infty} D_{n,q}^{a_n,b_n}(t^2;x) = x^2.$$

This complete the proof.

We recall the weighted spaces of the functions on $\mathbb{R}^+$, which are defined as follows:

$$P_\rho(\mathbb{R}^+) = \left\{f: |f(x)| \leq M_f \rho(x)\right\},$$
$$Q_\rho(\mathbb{R}^+) = \left\{f: f \in P_\rho(\mathbb{R}^+) \cap C[0,\infty)\right\},$$
$$Q_\rho^k(\mathbb{R}^+) = \left\{f: f \in Q_\rho(\mathbb{R}^+)\ and \lim_{x\to\infty}\frac{f(x)}{\rho(x)} = k(k\ is\ a\ constant\ )\right\},$$

where $\rho(x) = 1 + x^2$ is a weight function and M_f is a constant depending only on f. Note that $Q_\rho(\mathbb{R}^+)$ is a normed space with the norm $\| f \|_\rho = \sup_{x\geq 0}\frac{|f(x)|}{\rho(x)}$.

4. Rate of Convergence

Let $f \in C_B[0,\infty]$, the space of all bounded and continuous functions on $[0,\infty)$ and $x \geq \frac{1}{2n}, n \in \mathbb{N}$. Then for $\delta > 0$, the modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by

$$\omega(f,\delta) = \sup_{|t-x|\leq\delta} |f(t) - f(x)|, t \in 0,\infty). \tag{4.1}$$

It is known that $\lim_{\delta\to 0+}\omega(f,\delta) = 0$ for $f \in C_B[0,\infty)$ and for any $\delta > 0$ we have

$$|f(t) - f(x)| \leq \left(\frac{|t-x|}{\delta} + 1\right)\omega(f,\delta). \tag{4.2}$$

Now we calculate the rate of convergence of operators (2.2) by means of modulus of continuity and Lipschitz type maximal functions.

Theorem 4.1 *Let $D_{n,q}^{a_n,b_n}(.;.)$ be the operators defined by (2.2). Then for $f \in C_B[0,\infty), x \geq \frac{1}{2n}$ and $n \in \mathbb{N}$ we have*

$$|D_{n,q}^{a_n,b_n}(f;x) - f(x)| \leq 2\omega(f;\delta_{n,x}),$$

where

$$\delta_{n,x} = \sqrt{\left(\frac{a_n}{b_n} - 1\right)^2 x^2 + \left(\frac{a_n}{b_n}\right)[1+2\mu]_q \frac{x}{[n]_q}}. \tag{4.3}$$

Proof. We prove it by using (4.1), (4.2) and Cauchy-Schwarz inequality. We can easily get

$$|D_{n,q}^{a_n,b_n}(f;x) - f(x)| \leq \left\{1 + \frac{1}{\delta}\left(D_{n,q}^{a_n,b_n}(t-x)^2;x\right)^{\frac{1}{2}}\right\}\omega(f;\delta)$$

if we choose $\delta = \delta_{n,x}$ and by applying the result (2.2) of Lemma 2.2, we get the result.

Remark 4.2 *For every $f \in C_B[0,\infty), x \geq 0$ andn $\in \mathbb{N}$, suppose $\{a_n\},\{b_n\}$ be the sequence satisfies (2.1) then the operators $D_{n,q}^{*}(.;.)$ defined by (1.8) reduced to $D_{n,q}^{a_n,b_n}(.;.)$.*

Now we give the rate of convergence of the operators $D_{n,q}^{a_n,b_n}(f;x)$ defined in (2.2) in terms of the elements of the usual Lipschitz class $Lip_M(\nu)$.

Let $f \in C_B[0,\infty)$, $M > 0$ and $0 < \nu \leq 1$. The class $Lip_M(\nu)$ is defined as

$$Lip_M(\nu) = \{f: |f(\zeta_1) - f(\zeta_2)| \leq M|\zeta_1 - \zeta_2|^\nu (\zeta_1,\zeta_2 \in 0,\infty))\} \tag{4.4}$$

Theorem 4.3 *Let $D_{n,q}^{a_n,b_n}(.;.)$ be the operators defined in (2.2).Then for each $f \in Lip_M(\nu), (M > 0, 0 < \nu \leq 1)$ satisfying (4.4) we have*

$$|D_{n,q}^{a_n,b_n}(f;x) - f(x)| \leq M\left(\delta_{n,x}\right)^{\frac{\nu}{2}}$$

where $\delta_{n,x}$ is given in Theorem 4.1.

Proof. We prove it by using (4.4) and Hölder inequality. We have

$$\begin{aligned}|D_{n,q}^{a_n,b_n}(f;x) - f(x)| &\leq |D_{n,q}^{a_n,b_n}(f(t)-f(x);x)| \\ &\leq D_{n,q}^{a_n,b_n}(|f(t)-f(x)|;x) \\ &\leq M D_{n,q}^{a_n,b_n}(|t-x|^\nu;x).\end{aligned}$$

Therefore

$$\begin{aligned}|D_{n,q}^{a_n,b_n}(f;x) - f(x)| &\leq M\frac{[n]_q}{e_{\mu,q}(a_n[n]_q x)}\sum_{k=0}^{\infty}\frac{(a_n[n]_q x)^k}{\gamma_{\mu,q}(k)}\left|\frac{1-q^{2\mu\theta_k+k}}{b_n(1-q^n)} - x\right|^\nu \\ &\leq M\frac{[n]_q}{e_{\mu,q}(a_n[n]_q x)}\sum_{k=0}^{\infty}\left(\frac{(a_n[n]_q x)^k}{\gamma_{\mu,q}(k)}\right)^{\frac{2-\nu}{2}} \\ &\times\left(\frac{(a_n[n]_q x)^k}{\gamma_{\mu,q}(k)}\right)^{\frac{\nu}{2}}\left|\frac{1-q^{2\mu\theta_k+k}}{b_n(1-q^n)} - x\right|^\nu \\ &\leq M\left(\frac{n}{e_{\mu,q}(a_n[n]_q x)}\sum_{k=0}^{\infty}\frac{(a_n[n]_q x)^k}{\gamma_{\mu,q}(k)}\right)^{\frac{2-\nu}{2}} \\ &\times\left(\frac{[n]_q}{e_{\mu,q}(a_n[n]_q x)}\sum_{k=0}^{\infty}\frac{(a_n[n]_q x)^k}{\gamma_{\mu,q}(k)}\left|\frac{1-q^{2\mu\theta_k+k}}{b_n(1-q^n)} - x\right|^2\right)^{\frac{\nu}{2}}\end{aligned}$$

$$= M\left(D_{n,q}^{a_n,b_n}(t-x)^2;x\right)^{\frac{\nu}{2}}.$$

This completes the proof.
Let

$$C_B^2(\mathbb{R}^+) = \{g \in C_B(\mathbb{R}^+): g', g'' \in C_B(\mathbb{R}^+)\}, \tag{4.5}$$

with the norm

$$\| g \|_{C_B^2(\mathbb{R}^+)} = \| g \|_{C_B(\mathbb{R}^+)} + \| g' \|_{C_B(\mathbb{R}^+)} + \| g'' \|_{C_B(\mathbb{R}^+)}, \tag{4.6}$$

also

$$\| g \|_{C_B(\mathbb{R}^+)} = \sup_{x\in\mathbb{R}^+} |g(x)|. \tag{4.7}$$

Theorem 4.4 *Let $D_{n,q}^{a_n,b_n}(.;.)$ be the operators defined in (2.2). Then for any $g \in C_B^2(\mathbb{R}^+)$ we have*

$$|D_{n,q}^{a_n,b_n}(f;x) - f(x)| \leq \left\{\left(\left(\frac{a_n}{b_n}-1\right)x\right) + \frac{\delta_{n,x}}{2}\right\} \| g \|_{C_B^2(\mathbb{R}^+)},$$

where $\delta_{n,x}$ is given in Theorem 4.1.
Proof. Let $g \in C_B^2(\mathbb{R}^+)$. Then by using the generalized mean value theorem in the Taylor series expansion we have

$$g(t) = g(x) + g'(x)(t-x) + g''(\psi)\frac{(t-x)^2}{2}, \psi \in (x,t).$$

By applying linearity property on $D_{n,q}^{a_n,b_n}$, we have

$$D_{n,q}^{a_n,b_n}(g,x) - g(x) = g'(x)D_{n,q}^{a_n,b_n}((t-x);x) + \frac{g''(\psi)}{2}D_{n,q}^{a_n,b_n}((t-x)^2;x),$$

which implies that

$$\left|D_{n,q}^{a_n,b_n}(g;x) - g(x)\right|$$
$$\leq \left(\left(\frac{a_n}{b_n}-1\right)x\right) \| g' \|_{C_B(\mathbb{R}^+)} + \left(\left(\frac{a_n}{b_n}-1\right)^2 x^2 + \left(\frac{a_n}{b_n}\right)[1+2\mu]_q \frac{x}{[n]_q}\right)\frac{\| g'' \|_{C_B(\mathbb{R}^+)}}{2}.$$

From (4.6) we have $\| g' \|_{C_B[0,\infty)} \leq \| g \|_{C_B^2[0,\infty)}$.

$$|D_{n,q}^{a_n,b_n}(g;x) - g(x)| \leq \left(\left(\frac{a_n}{b_n}-1\right)x\right) \|$$
$$g \|_{C_B^2(\mathbb{R}^+)} + \left(\left(\frac{a_n}{b_n}-1\right)^2 x^2 + \left(\frac{a_n}{b_n}\right)[1+2\mu]_q \frac{x}{[n]_q}\right)\frac{\| g \|_{C_B^2(\mathbb{R}^+)}}{2}.$$

The proof follows from (2) of Lemma 2.2.
The Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf_{C_B^2(\mathbb{R}^+)}\left\{\left(\| f-g \|_{C_B(\mathbb{R}^+)} + \delta \| g'' \|_{C_B^2(\mathbb{R}^+)}\right): g \in \mathcal{W}^2\right\}, \tag{4.8}$$

where

$$\mathcal{W}^2 = \{g \in C_B(\mathbb{R}^+): g', g'' \in C_B(\mathbb{R}^+)\}. \tag{4.9}$$

There exits a positive constant $C > 0$ such that $K_2(f,\delta) \leq C\omega_2(f,\delta^{\frac{1}{2}}), \delta > 0$, where the second order modulus of continuity is given by

$$\omega_2(f,\delta^{\frac{1}{2}}) = \sup_{0<h<\delta^{\frac{1}{2}}} \sup_{x\in\mathbb{R}^+} |f(x+2h) - 2f(x+h) + f(x)|. \tag{4.10}$$

Theorem 4.5 *For $x \geq \frac{1}{2n}, n \in \mathbb{N}$ and $f \in C_B(\mathbb{R}^+)$ we have*

$|D_{n,q}^{a_n,b_n}(f;x)-f(x)|$

$$\leq 2M\left\{\omega_2\left(f;\sqrt{\frac{\left(2\left(\frac{a_n}{b_n}-1\right)x\right)+\delta_{n,x}}{4}}\right)+\min\left(1,\frac{\left(2\left(\frac{a_n}{b_n}-1\right)x\right)+\delta_{n,x}}{4}\right)\parallel f\parallel_{C_B(\mathbb{R}^+)}\right\},$$

where M is a positive constant, $\delta_{n,x}$ is given in Theorem 4.3 and $\omega_2(f;\delta)$ is the second order modulus of continuity of the function f defined in (4.10).

Proof. We prove this by using the Theorem 4.4

$$|D_{n,q}^{a_n,b_n}(f;x)-f(x)|\leq|D_{n,q}^{a_n,b_n}(f-g;x)|+|D_{n,q}^{a_n,b_n}(g;x)-g(x)|+|f(x)-g(x)|$$

$$\leq 2\parallel f-g\parallel_{C_B(\mathbb{R}^+)}+\frac{\delta_{n,x}}{2}\parallel g\parallel_{C_B^2(\mathbb{R}^+)}+\left(\left(\frac{a_n}{b_n}-1\right)x\right)\parallel g\parallel_{C_B(\mathbb{R}^+)}$$

From (4.6) clearly we have $\parallel g\parallel_{C_B[0,\infty)}\leq\parallel g\parallel_{C_B^2[0,\infty)}$.

Therefore,

$$|D_{n,q}^{a_n,b_n}(f;x)-f(x)|\leq 2\left(\parallel f-g\parallel_{C_B(\mathbb{R}^+)}+\frac{\left(2\left(\frac{a_n}{b_n}-1\right)x\right)+\delta_{n,x}}{4}\parallel g\parallel_{C_B^2(\mathbb{R}^+)}\right),$$

where $\delta_{n,x}$ is given in Theorem 4.1.

By taking infimum over all $g\in C_B^2(\mathbb{R}^+)$ and by using (4.8), we get

$$\left|D_{n,q}^{a_n,b_n}(f;x)-f(x)\right|\leq 2K_2\left(f;\frac{\left(2\left(\frac{a_n}{b_n}-1\right)x\right)+\delta_{n,x}}{4}\right).$$

Now for an absolute constant $Q>0$ in [3] we use the relation

$$K_2(f;\delta)\leq Q\{\omega_2(f;\sqrt{\delta})+\min(1,\delta)\parallel f\parallel\}.$$

This complete the proof.

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