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# Existence of solutions of general vector variational inequality problem

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**Abstract:** In this paper, we prove a Minty-type lemma for a new class of general vector variational inequality problem in Banach spaces. Using this lemma and KKM-Fan Theorem, we prove an existence theorem for general vector variational inequality problem. Further, we prove an existence theorem without monotonicity condition. Furthermore, using minimax theorem and concept of escaping sequence, we prove some existence theorems for the general vector variational inequality problems. Our results generalize and unify the same well-known results for the vector variational inequality.

**Keywords:** Operator of type ql, variational inequality, Minty-type lemma, KKM mapping.
2010 Mathematics subject classifications: 49J30, 47H10, 47H17, 90C99.

## 1. Introduction

The concept of vector variational inequality was introduced by Giannessi [8] in a finite dimensional space. Chen and Yang [4] considered general vector variational inequalities and vector complementary problems in infinite dimensional spaces and Chen [2] considered vector variational inequalities with a variable ordering structure. Yang [11] studied inverse vector variational inequalities and its relations with vector optimization problem. Through the last twenty years of development, existence results of solutions for several kinds of vector variational inequalities have been derived and the vector variational inequality problem has found many of its applications in vector optimization, set-valued optimization, approximate analysis of vector optimization problems and vector network equilibrium problem. Because of these applications, the study of vector variational inequalities has attracted wide attention. Throughout this paper, unless is otherwise specified, we assume that $X$ is a real Banach space and $X^*$ is the topological dual of $X$. We denote by $\langle x^*, x\rangle$ the value of the linear and continuous functional $x^* \in X^*$ at $x \in X$. Consider the set $K \subseteq X$ and let $A: K \to X^*$ and $a: K \to X$ be two given operators. Let $Y$ be a real Banach space and $C: K \to 2^Y$ be a set-valued mapping such that for all $x \in K, C(x)$ is a closed, convex and pointed cone with apex at origin and with nonempty interior $intC(x)$.

In 2011, László [9] studied the so-called general variational inequality of Stampacchia type (in short, GVI) which consists in finding an element $x \in K$ such that

$$\langle A(x), a(y) - a(x)\rangle \geq 0, \forall y \in K.$$

Motivated by the work of László [9], we shall study the following general vector variational inequality problem (in short GVVI): Find $x \in K$ such that

$$\langle A(x), a(y) - a(x)\rangle \notin -intC(x), \forall y \in K. \tag{1.1}$$

We prove a Minty-type lemma for GVVI (1.1) in Banach spaces. Using this lemma and KKM-Fan Theorem, we prove an existence theorem for GVVI (1.1). Further, we prove an existence theorem without monotonicity condition. Furthermore, using minimax theorem and concept of escaping sequence, we prove some existence theorems for GVVI (1.1). Our results generalize and unify some well-known results for the vector variational inequality.

## 2. Preliminaries

We recall some concepts and results which are used in establishing the results for GVVI (1.1).

**Definition 2.1.** [7] Let $K$ be a subset of a topological vector space $X$. A set-valued mapping $T: K \to 2^X$ is called a Knaster-Kuratowski-Mazurkiewieg mapping (KKM mapping) if for each nonempty finite subset $\{x_1, x_2, \ldots, x_n\} \subset K$, we have $Co\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} T(x_i)$, where $Co(A)$ is the convex hull of $A$.

Let $X$ be a real linear space. For $x, y \in X$, let us denote by $[x, y] = \{z := (1-t)x + ty : t \in [0,1]\}$ the closed line segment with the endpoints $x$ respectively $y$. The open line segment with the end points $x$ respectively $y$ is defined by $(x, y) := [x, y] \setminus \{x, y\} = \{z := (1-t)x + ty : t \in (0,1)\}$.

**Definition 2.2**. [9] Let $X$ and $Y$ be two real linear spaces. An operator $a: K \subseteq X \to Y$ is said to be of type ql if for every $x, y \in K$ and every $z \in [x, y] \cap K, a(z) \in [a(x), a(y)]$. Further, $a$ is said to be of type strict ql if for every $x, y \in K, x \neq y$ and every $z \in (x, y) \cap K, a(z) \in (a(x), a(y))$.

**Proposition 2.1.** [9] Let $a: I \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Then $a$ is of type ql if and only if $a$ is monotonic (increasing or decreasing).

**Example 2.1.** The function $a: \mathbb{R} \to \mathbb{R}$ defined by

$$a(x) = \begin{cases} x^2, \text{if} x > 0; \\ 3x, \text{if} x \leq 0, \end{cases}$$

is of type ql as it is monotonic increasing.

**Definition 2.3.** Let $X$ be a real Banach space, $X^*$ be its topological dual and let $A: K \subseteq X \to X^*$ be an operator. We say that $A$ is

(i) monotone if for every $x, y \in K, \langle A(x) - A(y), x - y\rangle \geq 0$;

(ii) pseudomonotone if for every $x, y \in K, \langle A(x), x - y\rangle \geq 0$ implies $\langle A(y), x - y\rangle \geq 0$.

**Definition 2.4.** [10] Let $X$ be a real Banach space, $X^*$ be its topological dual and let $A: K \subseteq X \to X^*$ and $a: K \to X^*$ be given operators. We say that $A$ is
(i) monotone relative to $a$ if for every $x, y \in K, \langle A(x) - A(y), a(x) - a(y) \rangle \geq 0$;
(ii) $a$ -pseudomonotone if for every $x, y \in K, \langle A(x), a(y) - a(x) \rangle \geq 0$ implies $\langle A(y), a(y) - a(x) \rangle \geq 0$.

**Definition 2.5.** A mapping $a: X \to Y$ is said to be completely continuous if and only if the weak convergence of $\{x_n\}$ to $x$ in $X$ implies the strong convergence of $\{a(x_n)\}$ to $a(x)$ in $Y$.
Now, we define the following.

**Definition 2.6**. Let $X$ and $Y$ be two real linear spaces and $K \subseteq X$. Let $a, b: K \to X$ be any two operators. The operator $a$ is said to be of type ql relative to $b$ if for every $x, y \in K$ and every $z \in [x, y] \cap K, a(z) \in [b(x), a(y)]$. Further, $a$ is said to be of type strict ql relative to $b$ if for every $x, y \in K, x \neq y$ and every $z \in (x, y) \cap K, a(z) \in (b(x), a(y))$.
We introduce the following definition.

**Definition 2.7.** Let $X$ be a real Banach space, $X^*$ be its topological dual and let $A: K \subseteq X \to X^*$. Let $a, b: K \to X$ be any two operators. Let $C$ be a closed, convex and pointed cone in $Y$. Then $A$ is said to be $C(x)$-monotone relative to $a$ and $b$ if for all $x, y \in K, \langle A(y) - A(x), a(y) - b(x) \rangle \in C(x)$.

**Definition 2.8.** Let $X$ be a real Banach space, $X^*$ be its topological dual and let $A: K \subseteq X \to X^*$ be an operator. We say that $A$ is $v$-hemicontinuous on $K$, if for any $x, y, z \in K$ and $\lambda \in (0,1)$, the mapping $\lambda \to \langle A(x + \lambda(y - x)), z \rangle$ is continuous at $0^+$.

**Definition 2.9.** [1] Let $X$ be a topological space and $K$ be a subset of $X$ such that $K = \bigcup_{n=1}^{\infty} K_n$ where $\{K_n\}_{n=1}^{\infty}$ is an increasing (in the sense that $K_n \subseteq K_{n+1}$) sequence of nonempty compact sets. A sequence $\{x_n\}_{n=1}^{\infty}$ in $K$ is said to be an escaping sequence from $K$ (relative to $\{K_n\}_{n=1}^{\infty}$) iff for each $n = 1,2, \ldots.$ there exists $m > 0$ such that $x_k \notin K_n, \forall k \geq m$.

Now, we need the following lemma and theorems to prove our existence results.
**Lemma 2.1**. [3] Let $C$ be an ordering cone in $Y$. Then for any $a, b, c \in Y$,
(i) $a \notin b + C$ implies that $a + c \notin b + c + C$;
(ii) $a \notin b + \text{int}C$ implies that $a + c \notin b + c + \text{int}C$;
(iii) $a \notin b - C$ implies that $a + c \notin b + c - C$;
(iv) $a \notin b - \text{int}C$ implies that $a + c \notin b + c - \text{int}C$.

**Theorem 2.1**. (KKM-Fan Lemma [7]) Let $K$ be a subset of a topological vector space $X$ and let $T: K \to 2^X$ be a KKM mapping. If for each $x \in K, T(x)$ is closed and for at least one

$x \in K, T(x)$ is compact, then $\bigcap_{x\in K} T(x) \neq \emptyset$.

**Theorem 2.2**. [9] Let $X$ and $Y$ be two real linear spaces, let $K \subseteq X$ be convex and let $a: K \to Y$ be an operator of type ql. Then for every $n \in \mathbb{N}$, every $x_1, x_2, \ldots, x_n \in K$ and every $x \in \mathrm{Co}\{x_1, x_2, \ldots, x_n\}$, we have $a(x) \in \mathrm{Co}\{a(x_1), a(x_2), \ldots, a(x_n)\}$.

**Theorem 2.3.** [7] Let $K$ be a nonempty compact and convex set in a Hausdorff topological vector space $X$. Let $B$ be a subset of $K \times K$ having the following properties:
(i) for each $x \in K, (x, x) \in B$;
(ii) for each $x \in K$, the set $B_x = \{y \in K: (x, y) \in B\}$ is closed;
(iii) for each $y \in K$, the set $B_y = \{x \in K: (x, y) \notin B\}$ is convex.
Then there exists a point $y_0 \in K$ such that $K \times \{y_0\} \subset B$.

**Theorem 2.4**. [6, 5] Let $K$ be a nonempty convex subset of a Hausdorff topological vector space $X$. Let $T: K \to 2^X$ be a set-valued mapping such that
(i) for each $x \in K, T(x)$ is nonempty convex subset of $K$;
(ii) for each $y \in K, T^{-1}(y) = \{x \in K: y \in T(x)\}$ contains an open set $O_y$ which may be empty;
(iii) $\bigcup_{y\in K} O_y = K$;
(iv) there exists a nonempty compact and convex subset $K_1$ of $K$ and points $\{x_1, \ldots, x_n\}$ in $K$ such that

$$D = \bigcap_{x\in K_1} O_x^c \subset \bigcup_{i=1}^{n} O_{x_i},$$

where $O_x^c$ is the complement of $O_x$ in $K$. Then there exists a point $x_0 \in K$ such that $x_0 \in T(x_0)$.

## 3. Existence of the Solutions of General Vector Variational Inequalities

Now, we prove the Minty-type Lemma.

**Lemma 3.1**. Let $X$ and $Y$ be two Banach spaces and $C$ be a closed, convex and pointed cone in $Y$. Let $K$ be any convex subset of $X$ and $b: K \to X$ be a given mapping. Let $a: K \to X$ be a given operator which is of type ql relative to $b$. Let $A: K \to X^*$ be $C(x)$-monotone relative to $a$ and $b$ and $v$-hemicontinuous on $K$. Then the following are equivalent:
(i) $x \in K, \langle A(x), a(y) - b(x)\rangle \notin -intC(x), \forall y \in K$;
(ii) $x \in K, \langle A(y), a(y) - b(x)\rangle \notin -intC(x), \forall y \in K$.

**Proof.** Since $A$ is $C(x)$-monotone relative to $a$ and $b$, we have for all $x, y \in K$,

$$\langle A(y) - A(x), a(y) - b(x)\rangle \in C(x).$$

This implies that

$$\langle A(y), a(y) - b(x)\rangle \in \langle A(x), a(y) - b(x)\rangle + C(x).$$

Now, let for all $y \in K$,

$$\langle A(x), a(y) - b(x)\rangle \notin -intC(x).$$

Then by Lemma 2.1, we have for all $y \in K$,

$$\langle A(y), a(y) - b(x) \rangle \notin -intC(x).$$

Conversely, suppose that for all $y \in K$,

$$\langle A(y), a(y) - b(x) \rangle \notin -intC(x).$$

Since $K$ is convex, then for any $\lambda \in (0,1)$ and $x, y \in K$,

$$\langle A(\lambda y + (1-\lambda)x), a(\lambda y + (1-\lambda)x) - b(x) \rangle \notin -intC(x),$$

i.e.

$$\langle A(x + \lambda(y-x)), a(x + \lambda(y-x)) - b(x) \rangle \notin -intC(x).$$

Since $a$ is of type ql relative to $b$, we have for all $x, y \in K$,

$$a(x + \lambda(y-x)) \in [b(x), a(y)],$$

$$a\big(x + \lambda(y-x)\big) = (1-t)b(x) + ta(y), \text{for some } t \in (0,1),$$

i.e.

$$a(x + \lambda(y-x)) = b(x) + t(a(y) - b(x)).$$

Therefore,

$$\langle A(x + \lambda(y-x)), t(a(y) - b(x)) \rangle \notin -intC(x),$$

$$\langle A(x + \lambda(y-x)), a(y) - b(x) \rangle \notin -intC(x).$$

Since $A$ is $v$-hemicontinuous, then as $\lambda \longrightarrow 0^+$, we have for all $y \in K$,

$$\langle A(x), a(y) - b(x) \rangle \notin -intC(x).$$

This completes the proof.

**Corollary 3.1**. Let $X$ and $Y$ be two Banach spaces and $C$ be a closed, convex and pointed cone in $Y$. Let $K$ be any convex subset of $X$. Let $a: K \to X$ be a given operator which is of type ql. Let $A: K \to X^*$ be $C(x)$-monotone relative to $a$ and $v$-hemicontinuous on $K$. Then the following are equivalent:

(i) $x \in K, \langle A(x), a(y) - a(x) \rangle \notin -intC(x), \forall y \in K$;

(ii) $x \in K, \langle A(y), a(y) - a(x) \rangle \notin -intC(x), \forall y \in K$.

Proof. The corollary can be proved by taking $a = b$ in Lemma 3.1.

To prove the next lemma, we set for any $y \in K$,

$$F_1(y) = \{x \in K: \langle A(x), a(y) - b(x) \rangle \notin -intC(x)\},$$

and

$$F_2(y) = \{x \in K: \langle A(y), a(y) - b(x) \rangle \notin -intC(x)\}.$$

Now, we equip $X$ with a weak topology, $Y$ with a strong topology and $X^*$ with the strong operator topology.

Now, we prove the following lemma.

**Lemma 3.2**. Let $X$ be a Banach space and $K \subset X$ be weakly compact. Let $A: K \to X^*$ be a vector valued function and let for $y \in K, A(y)$ be a completely continuous operator. Let $a: K \to X$ be a given operator and $b: K \to X$ be upper semicontinuous. Let the set-valued function $W: K \to 2^Y$ with $W(x) = Y \backslash (-intC(x))$ for every $x \in K$ be upper semicontinuous on $K$. Then $F_2(y)$ is weakly closed for every $y \in K$.

**Proof.** Let $\overline{F_2(y)}$ be the weakly closed hull of $F_2(y)$. Then there exists a sequence

$\{x_k\}_{k\in\mathbb{N}} \subset F_2(y)$ which converges weakly to some $x \in K$. Then, for every $y \in K$, we have
$$\langle A(y), a(y) - b(x_k)\rangle \notin -intC(x_k),$$
or
$$\langle A(y), a(y) - b(x_k)\rangle \in Y\backslash(-intC(x_k)).$$
This means that
$$\langle A(y), a(y) - b(x_k)\rangle \in W(x_k)$$
Since $A(y)$ is completely continuous then by upper semicontinuity of $W$ and $b$, we have
$$\langle A(y), a(y) - b(x)\rangle \in W(x).$$
This implies that
$$\langle A(y), a(y) - b(x)\rangle \notin -intC(x).$$
Hence, $x \in F_2(y)$, proving that $F_2(y)$ is weakly closed for every $y \in K$.

Now, we give the existence of solution of GVVI (1.1).

**Theorem 3.1**. Let $X$ and $Y$ be Banach spaces and $K \subset X$ be nonempty, convex and weakly compact. Let $C: X \to 2^Y$ be a set-valued mapping such that for every $x \in X, C(x)$ is a closed, convex and pointed cone with non empty interior int$C(x)$. Let the operator $a: K \to X$ be of the type ql which is upper semicontinuous. Let $A: K \to X^*$ be $C(x)$-monotone relative to $a$ and $v$-hemicontinuous on $K$ and let for every $y, A(y)$ be completely continuous operator. Let $W: K \to 2^Y$ be a set-valued mapping with $W(x) = Y\backslash(-intC(x))$ for every $x \in K$ which is upper semicontinuous on $K$. Then GVVI (1.1) admits solution.

**Proof**. Define the set-valued mappings $F_1, F_2: K \to 2^K$ by
$$F_1(y) = \{x \in K: \langle A(x), a(y) - a(x)\rangle \notin -intC(x)\},$$
and
$$F_2(y) = \{x \in K: \langle A(y), a(y) - a(x)\rangle \notin -intC(x)\}$$
respectively. Now, we show that $F_1$ is a KKM mapping on $K$. Consider
$$x_1, x_2, \ldots, x_n \in K, \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0, i = 1,2,\ldots,n$$
Let
$$x = \sum_{i=1}^{n} \alpha_i x_i$$
We show that $x \notin F_1(x_i)$, for $i = 1,2,\ldots,n$. Let us suppose on contrary $x \in F_1(x_i)$, for some $i$. Then
$$\langle A(x), a(x_i) - a(x)\rangle \in -intC(x), \text{for} i = 1,2,\ldots,n. \quad (3.1)$$

Since $a$ is of type ql, then by Theorem 2.2, we have
$$a(x) \in \mathrm{C}o\{a(x_1), a(x_2), \ldots, a(x_n)\}.$$
This implies that

$$a(x) = \sum_{i=1}^{n} \alpha_i a(x_i).$$

On multiplying the inequalities in (3.1), one by one with $\alpha_i$ and then adding, we have

$$\langle A(x), \sum_{i=1}^{n} \alpha_i a(x_i) - a(x)\rangle \in -intC(x),$$

$$\langle A(x), a(x) - a(x)\rangle \in -intC(x),$$

which is contradiction. Therefore

$$\mathrm{C}o\{a(x_1), a(x_2), \ldots, a(x_n)\} \subset \bigcup_{i=1}^{n} F_1(x_i).$$

Hence, $F_1$ is a KKM mapping on $K$.

Now, we show that $F_1(y) \subset F_2(y)$, for $y \in K$. For this, let $x \in F_1(y)$. Then

$$\langle A(x), a(y) - a(x)\rangle \notin -intC(x).$$

Now, $A$ is $C(x)$-monotone on $K$, we have

$$\langle A(y) - A(x), a(y) - a(x)\rangle \in C(x).$$

This implies that

$$\langle A(y), a(y) - a(x)\rangle \in \langle A(x), a(y) - a(x)\rangle + C(x).$$

By Lemma 2.1, we obtain

$$\langle A(y), a(y) - a(x)\rangle \notin -intC(x).$$

Therefore, $x \in F_2(y)$. Hence, $F_1(y) \subset F_2(y)$. This means that $F_2$ is also a KKM mapping.

By using Corollary 3.1, we have

$$\bigcap_{y\in K} F_1(y) = \bigcap_{y\in K} F_2(y).$$

Now, by Lemma 3.2, $F_2(y)$ is weakly closed for every $y \in K$. Since $K$ is weakly compact and $F_2(y) \subset K$ is weakly closed, we have that $F_2(y)$ is weakly compact. If we equip $X$ with weak topology, we can use the KKM Theorem for $F_2$. This, in turn, implies that

$$\bigcap_{y\in K} F_2(y) \neq \emptyset.$$

Therefore,

$$\bigcap_{y\in K} F_1(y) \neq \emptyset.$$

Then, there exists $x_0 \in K$ such that $x_0 \in \bigcap_{y\in K} F_1(y)$. Hence, for all $y \in K$,

$$\langle A(x_0), a(y) - a(x_0)\rangle \notin -intC(x_0).$$

This completes the proof.

**Corollary 3.2**. Let $X$ be a reflexive Banach space and $Y$ be a Banach space. Let $K \subset X$ be nonempty bounded, closed and convex. Let $C: X \to 2^Y$ be a set-valued mapping such that for every $x \in X, C(x)$ is a closed, convex and pointed cone with non-empty interior int$C(x)$. Let the operator $a: K \to X$ be of the type ql which is upper semicontinuous. Let $A: K \to X^*$ be $C(x)$-monotone relative to $a$ and $v$-hemicontinuous on $K$ and let for every $y, A(y)$ be completely continuous operator. Let $W: K \to 2^Y$ be a set-valued mapping with $W(x) = Y \backslash (-intC(x))$ for every $x \in K$ which is upper semicontinuous on $K$. Then GVVI (1.1) admits solution.

**Proof.** Since a bounded, closed and convex subset of a reflexive Banach space is weakly compact, we obtain that $K$ is weakly compact. Now, the proof is similar to that of Theorem 3.1.

Next, we give the following theorem.

**Theorem 3.2.** Let $X$ be a reflexive Banach space and $Y$ be a Banach space. Let $C: K \to 2^Y$ be a set-valued mapping such that for all $x \in K$, $C(x)$ is a closed, convex and pointed cone with nonempty interior $intC(x)$. Let $K$ be any nonempty closed, bounded and convex subset of $X$ with $0 \in K$. Let $A: K \to X^*$ be $C(x)$-monotone relative to $a$, $v$-hemicontinuous and let for every $y, A(y)$ be completely continuous operator. Let the operator $a: K \to X$ be of the type ql which is upper semicontinuous. If there exists some $r > 0$ such that for all $y, z \in K$,

$$\langle A(z), a(y) - a(0) \rangle \in intC(z), \tag{3.2}$$

with $\| y \| = r$, then there exists $\bar{x} \in K$ such that for all $y \in K$,

$$\langle A(\bar{x}), a(y) - a(\bar{x}) \rangle \notin -intC(\bar{x}).$$

**Proof**. Let $B_r = \{x \in X: \| x \| \leq r\}$. As $K$ and $B_r$ are closed and bounded subsets of the reflexive Banach space $X$, we have that $K \cap B_r$ is weakly compact. Then by Corollary 3.2, there exists $x_r \in K \cap B_r$ such that for all $y \in K \cap B_r$,

$$\langle A(x_r), a(y) - a(x_r) \rangle \notin -intC(x_r).$$

On putting $y = 0$, we have

$$\langle A(x_r), a(0) - a(x_r) \rangle \notin -intC(x_r),$$

i.e.

$$\langle A(x_r), a(0) - a(x_r) \rangle \in Y \backslash (-intC(x_r)). \tag{3.3}$$

Condition (3.2) implies that

$$\langle A(x_r), a(y) - a(0) \rangle \in intC(x_r). \tag{3.4}$$

On combining (3.3) and (3.4), we obtain

$$\langle A(x_r), a(y) - a(0) \rangle + \langle A(x_r), a(0) - a(x_r) \rangle \in intC(x_r) + Y \backslash (-intC(x_r)).$$

This implies that

$$\langle A(x_r), a(y) - a(x_r) \rangle \in intC(x_r) + Y \backslash (-intC(x_r)).$$

Since $intC(x_r) + Y \backslash (-intC(x_r)) \subseteq Y \backslash (-intC(x_r))$, this can be written as

$$\langle A(x_r), a(y) - a(x_r) \rangle \in Y \backslash (-intC(x_r)).$$

This implies that

$$\langle A(x_r), a(y) - a(x_r) \rangle \notin -intC(x_r).$$

Since $a$ is of type ql and $\| x \| \leq r$, then for any $z \in K$ and $\lambda \in (0,1)$ sufficiently small, we can write

$$a(y) = (1-\lambda)a(x_r) + \lambda a(z).$$

Therefore

$$\langle A(x_r), (1-\lambda)a(x_r) + \lambda a(z) - a(x_r)\rangle \notin -intC(x_r).$$

This implies that

$$\langle A(x_r), \lambda(a(z) - a(x_r))\rangle \notin -intC(x_r),$$

i.e.

$$\lambda\langle A(x_r), a(z) - a(x_r)\rangle \notin -intC(x_r),$$

or

$$\langle A(x_r), a(z) - a(x_r)\rangle \notin -intC(x_r).$$

This completes the proof.

Now, we prove the following theorem in more general setting.

**Theorem 3.3.** Let $X$ be a Hausdorff topological vector space and $Y$ be a Banach space. Let $K$ be a nonempty compact and convex subset of $X$. Let $A: K \to X^*$ be a nonlinear upper semicontinuous mapping. Let $C: X \to 2^Y$ be a set-valued upper semicontinuous mapping such that for every $x \in X, C(x)$ is a closed, convex and pointed cone with nonempty interior $intC(x)$. Let the operator $a: K \to X$ be of type ql which is upper semicontinuous. Then GVVI (1.1) has a solution.

**Proof.** Let assume for every $y, z \in K$,

$$B = \{(y,z) \in K \times K: \langle A(z), a(y) - a(z)\rangle \notin -intC(z)\}.$$

This implies that

$$(y,y) \in B, for every y \in K.$$

Now, define a set

$$B_y = \{z \in K: \langle A(z), a(y) - a(z)\rangle \notin -intC(z), \forall y \in K\}.$$

We show that $B_y$ is a closed set. Take any sequence $\{z_n\}_{n\in\mathbb{N}}$ in $B_y$ converging to $z$. Then for every $y \in K$, we have

$$\langle A(z_n), a(y) - a(z_n)\rangle \notin -intC(z_n).$$

Now, by the upper semi continuity of $A$, $a$ and $C$, we have for every $y \in K$,

$$\langle A(z), a(y) - a(z)\rangle \notin -intC(z).$$

This means that $z \in B_y$, proving that $B_y$ is closed.

Next, we define a set

$$B_z = \{y \in K: \langle A(z), a(y) - a(z)\rangle \in -intC(z), \forall z \in K\}.$$

We show that $B_z$ is convex.

For this, let $y_1, y_2 \in B_z$ and $\lambda \in (0,1)$. As $y_1, y_2 \in K$, then $\lambda y_1 + (1-\lambda)y_2 \in K$.

Since $a$ is type ql, we have

$$\langle A(z), a(\lambda y_1 + (1-\lambda)y_2) - a(z)\rangle = \langle A(z), ta(y_1) + (1-t)a(y_2) - a(z)\rangle, \text{for some } t \in (0,1),$$

$$= t\langle A(z), a(y_1) - a(z)\rangle + (1-t)\langle A(z), a(y_2) - a(z)\rangle, \text{for some } t \in (0,1),$$
$$\in -t intC(z) - (1-t) intC(z) = -intC(z),$$

i.e.

$$\langle A(z), a(\lambda y_1 + (1-\lambda)y_2) - a(z)\rangle \in -intC(z).$$

This implies that

$$\lambda y_1 + (1-\lambda)y_2 \in B_z.$$

Hence, $B_z$ is a convex set for every $z \in K$.

Now, by using Theorem 2.3, there exists a point $x_0 \in K$ such that $K \times \{x_0\} \subset B$ i.e. there exists $x_0 \in K$ such that

$$\langle A(x_0), a(y) - a(x_0)\rangle \notin -intC(x_0),$$

which completes the proof.

Now, we prove the following theorem.

**Theorem 3.4.** Let $K$ be a nonempty closed and convex subset of a Hausdorff topological vector space $X$. Let $Y$ be an ordered topological vector space and let $C: K \to 2^Y$ be a set-valued mapping such that for all $x \in K, C(x)$ is a closed, convex and pointed cone with nonempty interior $intC(x)$. Let the operator $a: K \to X$ be of type ql which is upper semicontinuous. Let $A: K \to X^*$ is $C(x)$-monotone relative to $a$ and $v$-hemicontinuous on $K$. For each $y \in K$, define

$$F_1(y) = \overline{\{x \in K: \langle A(x), a(y) - a(x)\rangle \notin -intC(x)\}},$$
$$F_2(y) = \{x \in K: \langle A(y), a(y) - a(x)\rangle \notin -intC(x)\}.$$

Further assume that there exists a nonempty compact and convex set $K_1 \subset K$ such that the following condition is satisfied:

There exist points $v_1, v_2, \ldots, v_n \in k$ with

$$\bigcap_{y \in K_1} F_1(y) \subset \bigcup_{i=1}^{n} (F_1(v_i))^c. \quad (3.5)$$

Then the solution set $S$ of GVVI (1.1) is nonempty.

**Proof.** We show that $F_2(y)$ is closed for each $y \in K$. Take any sequence $\{z_n\}_{n\in\mathbb{N}}$ in $F_2(y)$ such that $z_n \to z_0$. Then for each $y \in K$,

$$\langle A(y), a(y) - a(z_n)\rangle \notin -intC(x).$$

This implies that

$$\langle A(y), a(y) - a(z_n)\rangle \in Y\backslash(-intC(x)).$$

Since $Y\backslash(-intC)$ is closed and $a$ is upper semicontinuous, we have

$$\langle A(y), a(y) - a(z_0)\rangle \in Y\backslash(-intC(x)),$$

i.e.

$$\langle A(y), a(y) - a(z_0)\rangle \notin -intC(x),$$

which implies that $z_0 \in F_2(y)$ showing that $F_2(y)$ is closed.

Next, we show that $F_1(y) \subset F_2(y)$ for each $y \in K$. Take any $x \in F_1(y)$. Therefore

$$\langle A(x), a(y) - a(x)\rangle \notin -intC(x).$$

Since $A$ is $C$-monotone on $K$, we have for each $x, y \in K$,

$$\langle A(y) - A(x), a(y) - a(x)\rangle \in C(x).$$

This implies that

$$\langle A(y), a(y) - a(x)\rangle \in \langle A(x), a(y) - a(x)\rangle + C(x).$$

Therefore

$$\langle A(y), a(y) - a(x)\rangle \notin -intC(x).$$

Then $x \in F_2(y)$ proving that $F_1(y) \subset F_2(y)$ for each $y \in K$.

Now, suppose that condition (3.5) is satisfied. Consider the following condition

$$\langle A(x), a(x) - a(y)\rangle \in intC(x), \tag{3.6}$$

which may or may not hold. We prove the existence of solution in either case. Suppose that condition (3.6) does not hold. Then, there exists $x_0 \in K$ such that for all $y \in K$,

$$\langle A(x_0), a(x_0) - a(y)\rangle \notin intC(x),$$

which, in turn, implies that

$$\langle A(x_0), a(y) - a(x_0)\rangle \notin -intC(x).$$

Then $x_0 \in S$, the solution set of GVVI (1.1).

Now, suppose that condition (3.6) holds. In view of the Theorem 2.4, it is sufficient to prove that there exists $x_0 \in K$ such that

$$\langle A(x_0), a(y) - a(x_0)\rangle \notin -intC(x). \; (3.7)$$

If there is no solution to the above problem (3.7), then for all $x \in K$, we have

$$\langle A(x), a(y) - a(x)\rangle \in -intC(x),$$

i.e.

$$\langle A(x), a(x) - a(y)\rangle \in intC(x).$$

Now, define a set-valued mapping $T: K \to 2^K$ by

$$T(x) = \{y \in K: \langle A(x), a(y) - a(x)\rangle \in -intC(x)\}.$$

Clearly, $T(x)$ is nonempty. Now, we prove that $T(x)$ is convex for all $x \in K$. Let $y_1, y_2 \in T(x)$ and $\lambda \in (0,1)$. Since $K$ is convex and $T(x) \subseteq K$, then $\lambda y_1 + (1-\lambda)y_2 \in K$. Now, as $a$ is of type ql, we have

$$\langle A(x), a(\lambda y_1 + (1-\lambda)y_2) - a(x)\rangle = \langle A(x), ta(y_1) + (1-t)a(y_2) - a(x)\rangle, \text{for some } t \in (0,1),$$

$$= t\langle A(x), a(y_1) - a(x)\rangle + (1-t)\langle A(x), a(y_2) - a(x)\rangle, \text{for some } t \in (0,1),$$

$$\in -t\, intC(x) - (1-t)intC(x) = -intC(x).$$

This implies that

$$\lambda y_1 + (1-\lambda)y_2 \in T(x).$$

Hence, $T(x)$ is a convex set for all $x \in K$.

Further, for each $y \in K$,

$T^{-1}(y) = \{x \in K: y \in T(x)\}$,

$= \{x \in K: \langle A(x), a(y) - a(x)\rangle \in -intC(x)\}$,

$= \{x \in K: \langle A(x), a(y) - a(x)\rangle \notin -intC(x)\}^c$,

$\supset (\overline{\{x \in K: \langle A(x), a(y) - a(x)\rangle \notin -intC(x)\}})^c$,

$= F_1(y)^c \equiv O(y),$
which is relatively open in $K$.
Since, condition (3.6) holds then for each $y \in K$, there exists $x \in K$ such that

$$x \in F_2(y)^c \subset F_1(y)^c,$$

and hence

$$\bigcup_{y \in K} F_1(y)^c = \bigcup_{y \in K} O(y) = K.$$

Now, from condition (3.5), there exists points $v_1, v_2, \dots, v_n \in K$ such that

$$\bigcap_{y \in K_1} O(y)^c \subset \bigcup_{i=1}^{n} O(v_i).$$

Thus, all the assumptions of Theorem 2.4 are satisfied. Hence there exists $x_0 \in T(x)$, i.e.

$$\langle A(x_0), a(x_0) - a(x_0)\rangle \in -intC(x),$$

which is a contradiction.

Hence, the problem (3.7) has a solution and the theorem is proved.
Next, using the concept of escaping sequence, we show the existence of solution.

**Theorem 3.5.** Let $X$ be a Hausdorff topological vector space, let $K$ be a subset of $X$ such that $K = \bigcup_{n=1}^{\infty} K_n$ where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty, compact and convex subsets of $K$ and let $Y$ be a regular topological vector space. Let $A: K \to X^*$ be $C(x)$-monotone relative to $a$, $v$-hemicontinuous and completely continuous and let $a: K \to X$ be of the type ql which is upper semicontinuous on $K$. Let $C: K \to 2^Y$ be a set-valued mapping such that for all $x \in K$, $C(x)$ is a closed, convex and pointed cone with nonempty interior $intC(x)$ and let $W: K \to 2^Y$, defined by $W(x) = Y \backslash (-intC(x))$, be upper semicontinuous. Further, let for each sequence $\{x_n\}_{n=1}^{\infty}$ in $K$ with $x_n \in K_n, \forall n = 1,2,3, \dots$, which is escaping from $K$ relative to $\{K_n\}_{n=1}^{\infty}$, there exists $m \in \mathbb{N}$ and $z_m \in K_m$ such that

$$\langle A(x_m), a(z_m) - a(x_m)\rangle \in -intC(x_m). \qquad (3.8)$$

Then there exists $x_0 \in K$ such that for all $y \in K$,

$$\langle A(x_0), a(y) - a(x_0)\rangle \notin -intC(x_0).$$

**Proof.** By using Theorem 3.3, for all $n \in \mathbb{N}$, there exists $x_n \in K_n$ such that for all $z \in K_n$, we have

$$\langle A(x_n), a(z) - a(x_n)\rangle \notin -intC(x_n). \qquad (3.9)$$

Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ be escaping from $K$ relative to $\{K_n\}_{n=1}^{\infty}$. Then by the given condition (3.8), there exists $z_m \in K_m$ such that

$$\langle A(x_m), a(z_m) - a(x_m)\rangle \in -intC(x_m),$$

which contradicts (3.9). Therefore, $\{x_n\}_{n=1}^{\infty}$ is not an escaping sequence from $K$ relative to $\{K_n\}_{n=1}^{\infty}$. Then there exists $p \in \mathbb{N}$ and there is some subsequence $\{x_{j_n}\}$ of $\{x_n\}_{n=1}^{\infty}$ which must lie entirely in $K_p$. Since $K_p$ is compact, there is a subsequence $\{x_{i_n}\}_{i_n\in\Gamma}$ of $\{x_{j_n}\}$ in $K_p$ and there exists $x\in K_p$ such that $x_{i_n}\to x$ where $i_n\to\infty$. Since $\{K_n\}_{n=1}^{\infty}$ is increasing, for all $y\in K$ there exists $i_0\in\Gamma$ with $i_0>p$ such that $y\in K_{i_0}$ and for all $i_n\in\Gamma$ and $i_n>i_0$, we have $y\in K_{i_0}\subseteq K_{i_n}$ such that

$$\langle A(x_{i_n}), a(y)-a(x_{i_n})\rangle \notin -intC(x_{i_n}).$$

This implies that

$$\langle A(x_{i_n}), a(y)-a(x_{i_n})\rangle \in Y\backslash(-intC(x_{i_n})),$$

i.e.

$$\langle A(x_{i_n}), a(y)-a(x_{i_n})\rangle \in W(x_{i_n}).$$

Since $A$ is completely continuous, we may obtain that there exists $x_0\in K$ such that $A(x_{i_n})\to A(x_0)$ as $i_n\to\infty$. Now, by the continuity of inner product and semicontinuity of $a$, we have

$$\langle A(x_{i_n}), a(y)-a(x_{i_n})\rangle \to \langle A(x_0), a(y)-a(x_0)\rangle.$$

Since $W$ is upper semicontinuous and $Y$ be regular, we have

$$\langle A(x_0), a(y)-a(x_0)\rangle \in W(x_0).$$

This implies that

$$\langle A(x_0), a(y)-a(x_0)\rangle \notin -intC(x_0),$$

and this completes the proof.

**Example 3.1.** Let $X=Y=\mathbb{R}$, $K_n=[2,n+2]$, $n=1,2,3,\ldots$ and $K=\bigcup_{n=1}^{\infty}K_n$, where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty, compact and convex subsets of K. Let for all $x\in K$, $C(x)=[0,+\infty[$ be a closed, convex and pointed cone. Let for all $x\in K$, $A:K\to X^*$ be defined by $A(x)=x$ and $a:K\to X$ be such that $a(x)=2x$. Now, by using

**Theorem 3.3**. for all $n\in\mathbb{N}$, there exists $x_n=2\in K_n$ such that for all $z\in K_n$,

$$\langle A(x_n), a(z)-a(x_n)\rangle = x_n(2z-2x_n)=2(2z-4)\geq 0.$$

By condition (3.8) of Theorem 3.5, it can be easily seen that $\{x_n\}_{n=1}^{\infty}$ is not an escaping sequence from $K$ relative to $\{K_n\}_{n=1}^{\infty}$ and hence $x_0=2\in K$ is such that for all $y\in K$,

$$\langle A(x_0), a(y)-a(x_0)\rangle = x_0(2y-2x_0)=2(2y-4)\geq 0.$$

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# Warped Product and Doubly Warped Product Bi-slant Submanifolds in trans-Sasakian Manifolds

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**Abstract:** In order to construct Riemannian manifolds with negative sectional curvature, the notion of warped products was first defined by Bishop & O'Neill in [22]. In general, doubly warped products can be considered as generalization of warped products. In this paper, we give complete classification of a warped product bi-slant submanifold in a trans-Sasakian manifold under a geometric condition. In continuation, we prove the non-existence property for doubly warped product bi-slant submanifolds in trans-Sasakian manifolds. Moreover, we check the existence of doubly warped product bi-slant submanifolds in different ambient manifolds such as Sasakian, Kenmotsu and cosymplectic manifolds.

**Keywords:** warped products, doubly warped products, bi-slant submanifolds, trans-Sasakian manifolds.
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## 1. Introduction

The warped products were first defined by Bishop and O'Neill [22]. They used this concept to construct Riemannian manifolds with negative sectional curvature. In general, doubly warped products can be considered as a generalization of warped products. Beem and Powell considered these products for Lorentzian manifolds in [15]. Then Allison [11] considered causality and global hyperbolicity of doubly warped products and null pseudocovexity of Lorentzian doubly warped products in [12]. Conformal properties of doubly warped products are studied by Gebarowski [1]. B.Y. Chen [8, 9] was the first who initiated the study of warped product submanifolds by showing that there do not exist warped product CR-submanifolds of the type $\mathcal{M}_\perp \times_f \mathcal{M}_T$ and he considered warped product CR-submanifolds of the types $\mathcal{M}_T \times_f \mathcal{M}_\perp$ and established a relationship between the warping function $f$ and the squared norm of the second fundamental form. Later on, the geometrical aspect of warped products and doubly warped products had been studied by many researchers (for example [20, 23, 24, 25]).

The notion of bi-slant submanifolds was defined by A. Carriazo et al. [16] as a generalization of contact CR, slant and semi-slant submanifolds. Such submanifolds generalize invariant, anti-invariant and pseudo-slant submanifolds as well. Many articles on warped product submanifolds of trans-Sasakian manifolds are available in literature [19, 20, 26]. By generalizing the notion of warped product bi-slant submanifolds in trans-Sasakian manifolds, in the present paper, we wish to check the existence of other product bi-slant submanifolds such as doubly warped products in trans-Sasakian manifolds.

Our work is structured as follows: Section 2 is preliminary in nature. In this section, we present basic material about trans-Sasakian manifolds, warped products and bi-slant submanifolds. In Section 3, we give complete classification of a warped product bi-slant submanifold in a trans-Sasakian manifold (Theorems 1 and 2) with an example (Example 2). In Section 4, we check whether the doubly warped product bi-slant submanifolds in trans-Sasakian manifolds exist or not (Theorems 3 and 4). Sections 5 and 6 deal with the applications of the results (Theorems 3 and 4) obtained in Section 4.

## 2. Preliminaries

An odd dimensional differentiable manifold $\overline{\mathcal{M}}$ has an almost contact structure $(\phi,\xi,\eta,g)$ if there exists on $\overline{\mathcal{M}}$ a tensor field $\phi$ of type (1, 1), a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ such that [17]

$$\phi^2=-I+\eta\otimes\xi,\quad \phi\xi=0,\quad \eta(\xi)=1,\quad \eta(\phi)=0,\quad \eta(X)=g(X,\xi), \tag{1}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\quad g(\phi X,Y)+g(X,\phi Y)=0. \tag{2}$$

Here and in the sequel $X,Y,Z,\ldots$ always denote arbitrary vector fields on $\overline{\mathcal{M}}$. The fundamental 2-form $\varphi$ on $\overline{\mathcal{M}}$ is defined by

$$\varphi(X,Y)=g(\phi X,Y).$$

An almost contact metric structure $(\phi,\xi,\eta,g)$ on $\overline{\mathcal{M}}$ is called a trnas-Sasakian structure [14] if $(\overline{\mathcal{M}}\times\mathbb{R},J,G)$ belongs to the class $W_4$ of the Gray-Hervella classification of almost Hermitian manifolds [3], where $J$ is the almost complex structure on $\overline{\mathcal{M}}\times\mathbb{R}$ defined by

$$J(X,ad/dt)=(\phi X-a\xi,\eta(X)d/dt)$$

for all vector fields X on $\overline{\mathcal{M}}$ and smooth functions $a$ on $\overline{\mathcal{M}}\times\mathbb{R}$ and $G$ is the product metric on $\overline{\mathcal{M}}\times\mathbb{R}$. This may be expressed by the following condition:

$$(\overline{\nabla}_X\phi)Y=\alpha(g(X,Y)\xi-\eta(Y)X)+\beta(g(\phi X,Y)\xi-\eta(Y)\phi X) \tag{3}$$

for some smooth functions $\alpha$ and $\beta$ on $\overline{\mathcal{M}}$ and this trans-Sasakian structure is termed as structure of type $(\alpha,\beta)$.

A trans-Sasakian of type

1. $(\alpha,0)$ is $\alpha$ −Sasakian if $\beta=0$;
2. $(0,\beta)$ is $\beta$ −Kenmotsu $\alpha=0$;

A trans-Sasakian structure of type $(\alpha,\beta)$ is

1. Sasakian if $\beta = 0, \alpha = 1$;
2. Kenmotsu $\alpha = 0, \beta = 1$;
3. cosymplectic if $\alpha = \beta = 0$.

Let $\mathcal{M}$ be any submanifold in a Riemannian manifold $\overline{\mathcal{M}}$. We put dim $\mathcal{M} = n$ and dim $\overline{\mathcal{M}} = 2m + 1$. The Riemannian metric for $\mathcal{M}$ and $\overline{\mathcal{M}}$ is denoted by the same symbol $g$. Let $T\mathcal{M}$ and $T^{\perp}\mathcal{M}$ denote the Lie algebra of vector field and set of all normal vector fields on $\mathcal{M}$ respectively. The operator of covariant differentiation with respect to the Levi-Civita connection in $\mathcal{M}$ and $\overline{\mathcal{M}}$ is denoted by $\nabla$ and $\overline{\nabla}$, respectively. The Gauss and Weingarten formulas are, respectively, given as [17]

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4}$$

and

$$\overline{\nabla}_X V = -A_V(X) + \nabla^{\perp}_X V \tag{5}$$

for any vector fields $X, Y \in T\mathcal{M}$ and $V \in T^{\perp}\mathcal{M}$. Here $h$ is the second fundamental form, $A$ is the shape operator and $\nabla^{\perp}$ is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T^{\perp}\mathcal{M}$.

The second fundamental form and the shape operator are related as [17]

$$g(h(X, Y), V) = g(A_V(X), Y).$$

for any vector fields $X, Y \in T\mathcal{M}$ and $V \in T^{\perp}\mathcal{M}$. Here $g$ denote the induced metric on $\mathcal{M}$ as well as the Riemannian metric on $\overline{\mathcal{M}}$. Moreover, the covariant derivative of the tensor filed $\phi$ is defined as

$$(\overline{\nabla}_X \phi)Y = \overline{\nabla}_X \phi Y - \phi \overline{\nabla}_X Y. \tag{6}$$

Let $\wp \in \mathcal{M}$ and $\{\mathcal{E}_1, \dots, \mathcal{E}_n\}$ be a local orthonormal frame of $T_{\wp}\mathcal{M}$ and $\{\mathcal{E}_{n+1}, \dots, \mathcal{E}_{2m+1}\}$ be a local orthonormal frame of $T^{\perp}_{\wp}\mathcal{M}$. The mean curvature vector $\mathcal{H}$ of a submanifold $\mathcal{M}$ at $\wp$ is given by [17]

$$\mathcal{H} = \frac{1}{n} \sum_{i=1}^{n} h(\mathcal{E}_i, \mathcal{E}_i).$$

Also, we set

$$h_{ij}^r = g(h(\mathcal{E}_i, \mathcal{E}_j), \mathcal{E}_r), \quad i, j \in \{1, \dots, n\}, \quad r \in \{n + 1, \dots, 2m + 1\}$$

and

$$||h||^2 = \sum_{i,j=1}^{n} g(h(\mathcal{E}_i, \mathcal{E}_j), h(\mathcal{E}_i, \mathcal{E}_j)).$$

A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is said to be [17]

- totally umbilical if $h(X,Y) = g(X,Y)\mathcal{H}$ for any vector fields $X, Y \in T\mathcal{M}$;
- totally geodesic if $h(X,Y) = 0$ for any vector fields $X, Y \in T\mathcal{M}$;
- minimal if $\mathcal{H} = 0$, i.e., trace $h \equiv 0$.

For any vector field $X \in T\mathcal{M}$, we put [17]

$$\phi X = \mathcal{P}X + \mathcal{F}X, \tag{7}$$

where $\mathcal{P}X = tan(\phi X)$ and $\mathcal{F}X = nor(\phi X)$. Then $\mathcal{P}$ is an endomorphism of $T\mathcal{M}$, and $\mathcal{F}$ is the normal bundle valued $1-$form on $T\mathcal{M}$.

In the same way, for any vector field $V \in T^{\perp}\mathcal{M}$, we put [17]

$$\phi V = \mathcal{B}V + \mathcal{C}V, \tag{8}$$

where $\mathcal{B}V = tan(\phi V)$ and $\mathcal{C}V = nor(\phi V)$.

It is easy to see the following formulas:

$$g(\mathcal{P}X, Y) = -g(X, \mathcal{P}Y) \tag{9}$$

$$g(\mathcal{C}U, V) = -g(U, \mathcal{C}V) \tag{10}$$

$$g(\mathcal{F}X, V) = -g(X, \mathcal{B}V) \tag{11}$$

for any vector fields $X, Y \in T\mathcal{M}$ and $U, V \in T^{\perp}\mathcal{M}$. For other geometric relations, see [17]. Following are the different classes of submanifolds in trans-Sasakian manifolds:

**Definition 1.** A submanifold $\mathcal{M}$ of an almost contact metric manifold $\overline{\mathcal{M}}$ is said to be invariant if $\mathcal{F} \equiv 0$, that is, $\phi X \in T\mathcal{M}$, and anti-invariant if $\mathcal{P} \equiv 0$, that is, $\phi X \in T^{\perp}\mathcal{M}$, for any vector field $X \in T\mathcal{M}$.

In contact geometry, A. Lotta introduced slant immersions as follows [4]:

**Definition 2.** Let $\mathcal{M}$ be a submanifold of an almost contact metric manifold $\overline{\mathcal{M}}$. For each non-zero vector $X$ tangent to $\mathcal{M}$ at $p$, the angle $\theta(p) \in [0, \frac{\pi}{2}]$, between $\phi X$ and $\mathcal{P}X$ is called the slant angle of $\mathcal{M}$. If the slant angle is constant for each $X \in T\mathcal{M}$ and $p \in \mathcal{M}$, then the submanifold is called the slant submanifold.

For slant submanifolds, the following facts are known [21]:

$$\mathcal{P}^2(X) = cos^2\theta(-X + \eta(X)\xi), \tag{12}$$

$$g(\mathcal{P}X, \mathcal{P}Y) = cos^2\theta(g(X,Y) - \eta(Y)\eta(X)) \tag{13}$$

and

$$g(\mathcal{F}X, \mathcal{F}Y) = sin^2\theta(g(X,Y) - \eta(Y)\eta(X)) \tag{14}$$

for any vector fields $X, Y \in T\mathcal{M}$, where $\theta$ is the slant angle of $\mathcal{M}$.

There are some other important classes of submanifolds which are determined by the behavior of tangent bundle of the submanifold under the action of an almost contact metric structure $\phi$ of $\overline{\mathcal{M}}$ [25]:

1. A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called a contact CR-submanifold of $\overline{\mathcal{M}}$ if there exists a differentiable distribution $D$ on $\mathcal{M}$ whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant.
2. A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called semi-slant submanifold of $\overline{\mathcal{M}}$ if there exists a pair of orthogonal distributions $D$ and $D_\theta$ such that $D$ is invariant and $D_\theta$ is proper slant.
3. A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called pseudo-slant submanifold of $\overline{\mathcal{M}}$ if there exists a pair of orthogonal distributions $D^{\perp}$ and $D_\theta$ such that $D^{\perp}$ is anti-invariant and $D_\theta$ is proper slant.

**Definition 3.** [16] A submanifold $\mathcal{M}$ of an almost contact metric manifold $\overline{\mathcal{M}}$ is said to be a bi-slant submanifold if there exists a pair of orthogonal distributions $D_{\theta_1}$ and $D_{\theta_2}$ of $\mathcal{M}$ such that

1. $T\mathcal{M}$ admits the orthogonal direct decomposition: $T\mathcal{M} = D_{\theta_1} \oplus D_{\theta_2} \oplus \{\xi\}$;
2. Each distribution $D_{\theta_i}$ is slant with the slant angle $\theta_i$ for $i = 1,2$.

A bi-slant submanifold of an almost contact metric manifold $\overline{\mathcal{M}}$ is called proper if the slant distributions $D_{\theta_1}$ and $D_{\theta_2}$ are of the slant angles $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

If we assume

1. $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then $\mathcal{M}$ is a CR-submanifold;
2. $\theta_1 = 0$ and $\theta_2 \neq 0, \frac{\pi}{2}$, then $\mathcal{M}$ is a semi-slant submanifold;
3. $\theta_1 = \frac{\pi}{2}$ and $\theta_2 \neq 0, \frac{\pi}{2}$, then $\mathcal{M}$ is a pseudo-slant submanifold.

For a bi-slant submanifold $\mathcal{M}$ of an almost contact metric manifold, the normal bundle of $\mathcal{M}$ is decomposed as

$$T^{\perp}\mathcal{M} = \mathcal{F}D_{\theta_1} \oplus \mathcal{F}D_{\theta_2} \oplus \mu, \tag{15}$$

where $\mu$ is a $\phi$-invariant normal subbundle of $\mathcal{M}$.

## 3. Warped Product Bi-slant Submanifolds

**Definition 4.** [22] Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be two Riemannian manifolds and $f > 0$ be a differentiable function on $\mathcal{M}_1$. Consider the product $\rho: \mathcal{M}_1 \times \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ and $\delta: \mathcal{M}_1 \times \mathcal{M}_2 \longrightarrow \mathcal{M}_2$. The projection maps given by $\rho(p,q) = p$ and $\delta(p,q) = q$ for any $(p,q) \in \mathcal{M}_1 \times \mathcal{M}_2$. Then the warped product $\mathcal{M} = \mathcal{M}_1 \times_f \mathcal{M}_2$ is the product manifold $\mathcal{M}_1 \times \mathcal{M}_2$ equipped with the Riemannian structure such that

$$g(X,Y) = g_1(\rho^*X,\rho^*Y) + (fo\rho)^2 g_2(\delta^*X,\delta^*Y) \quad (16)$$

for any $X,Y \in T\mathcal{M}$, where $*$ is the symbol for the tangent maps, and we have $g = g_1 + f^2 g_2$. The function $f$ is called the warping function of $\mathcal{M}$.

In particular, a warped product manifold is said to be trivial if its warping function is constant. In such a case, we call the warped product manifold a Riemannian product manifold. If $\mathcal{M} = \mathcal{M}_1 \times_f \mathcal{M}_2$ is a warped product manifold then $\mathcal{M}_1$ is totally geodesic and $\mathcal{M}_2$ is totally umbilical submanifold of $\mathcal{M}$, respectively [22].

Let $\mathcal{M} = \mathcal{M}_1 \times_f \mathcal{M}_2$ be a warped product manifold with the warping function $f$. Then

$$\nabla_X Z = \nabla_Z X = (Xlnf)Z$$

for each $X \in T\mathcal{M}_1$ and $Z \in T\mathcal{M}_2$, where $Xlnf$ is the gradient of $lnf$ and $\nabla$ denotes the Levi-Civita connection on $\mathcal{M}$.

Example 1: The standard space-time models of the universe are warped products as the simplest models of neighbourhoods of stars and black holes.

Now, we define the notion of warped product bi-slant submanifolds in a trans-Sasakian manifold as follows:

**Definition 5.** A warped product $\mathcal{M}_1 \times_f \mathcal{M}_2$ of two slant submanifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ of a trans-Sasakian manifold $\overline{\mathcal{M}}$ is called a warped product bi-slant submanifold.

A warped product bi-slant submanifold $\mathcal{M}_1 \times_f \mathcal{M}_2$ is called proper if $\mathcal{M}_1$ and $\mathcal{M}_2$ are proper slant in $\overline{\mathcal{M}}$. Otherwise, the warped product bi-slant submanifold $\mathcal{M}_1 \times_f \mathcal{M}_2$ is called non-proper.

At this moment, we need the following lemma, which shall be required to prove our main results of this section:

**Lemma 1.** Let $\mathcal{M} = \mathcal{M}_1 \times_f \mathcal{M}_2$ be a warped product bi-slant submanifold with bi-slant angle $\{\theta_1,\theta_2\}$ in a trans-Sasakian manifold $\overline{\mathcal{M}}$. Then, for any $X_1 \in T\mathcal{M}_1$ and $X_2, Y_2 \in T\mathcal{M}_2$,

$$g(h(X_1,X_2),\mathcal{F}Y_2) = g(h(X_1,Y_2),\mathcal{F}X_2)$$

holds.

Proof. For any $X_1 \in T\mathcal{M}_1$ and $X_2, Y_2 \in T\mathcal{M}_2$, we have

$$\begin{aligned} g(h(X_1,X_2),\mathcal{F}Y_2) &= g(\overline{\nabla}_{X_1}X_2,\phi Y_2) - g(\nabla_{X_1}X_2,\mathcal{P}Y_2) \\ &= -g(\phi\overline{\nabla}_{X_1}X_2,Y_2) - (X_1 lnf)g(X_2,\mathcal{P}Y_2) \end{aligned} \quad (17)$$

$$= g((\overline{\nabla}_{X_1}\phi)X_2, Y_2) - g(\overline{\nabla}_{X_1}\phi X_2, Y_2) - (X_1 lnf)g(X_2, \mathcal{P}Y_2)$$
$$= g(A_{\mathcal{F}X_2}X_1, Y_2),$$

where we have used

$$(\overline{\nabla}_{X_1}\phi)X_2 = \alpha\{g(X_1, X_2)\xi - \eta(X_2X_1)\} + \beta\{g(\phi X_1, X_2)\xi - \eta(X_2)\phi X_1\}$$

Hence, we get our assertion from (17).

For a warped product bi-slant submanifold in a trans-Sasakian manifold such that $\xi \in T\mathcal{M}_1$, we have the following result:

**Theorem 4.1:** *Let $\mathcal{M} = \mathcal{M}_1 \times_f \mathcal{M}_2$ be a warped product bi-slant submanifold with bi-slant angle $\{\theta_1, \theta_2\}$ in a trans-Sasakian manifold $\overline{\mathcal{M}}$ such that $\xi \in T\mathcal{M}_1$. Then one of the following these cases must occur:*

1. *$\mathcal{M}$ is a warped product pseudo-slant submanifold such that $\mathcal{M}_2$ is a totally real submanifold $\mathcal{M}^\perp$ of $\overline{\mathcal{M}}$;*
2. *If $\overline{\mathcal{M}}$ is $\alpha$ −Sasakian manifold, i.e., $\beta = 0$, then $\mathcal{M}$ is a Riemannian product;*
3. *If $\beta \neq 0$, then $\beta\eta(X_1) = (X_1 lnf)$.*

**Proof.** For any $X_1 \in T\mathcal{M}_1$ and $X_2, Y_2 \in T\mathcal{M}_2$, we have

$$\begin{aligned} g(h(X_1, X_2), \mathcal{F}Y_2) &= g(\overline{\nabla}_{X_2}X_1, \phi Y_2) - g(\nabla_{X_2}X_1, \mathcal{P}Y_2) \\ &= -(\mathcal{P}X_1 lnf)g(X_2, Y_2) + g(h(X_2, Y_2), \mathcal{F}X_1) - \alpha\eta(X_1)g(X_2, Y_2) \\ &\quad -\beta\eta(X_1)g(\mathcal{P}X_2, Y_2) - (X_1 lnf)g(X_2, \mathcal{P}Y_2) \end{aligned} \tag{18}$$

Interchanging the role of $X_2$ by $Y_2$ in equation (18), we find that

$$\begin{aligned} g(h(X_1, Y_2), \mathcal{F}X_2) &= -g(\overline{\nabla}_{Y_2}\mathcal{P}X_1, X_2) - g(\overline{\nabla}_{Y_2}\mathcal{F}X_1, X_2) - \alpha\eta(X_1)g(Y_2, X_2) \\ &\quad -\beta\eta(X_1)g(\mathcal{P}Y_2, X_2) - (X_1 lnf)g(Y_2, \mathcal{P}X_2). \end{aligned} \tag{19}$$

Subtracting (19) from (18), we get

$$g(\mathcal{P}Y_2, X_2)[-(X_1 lnf) + \beta\eta(X_1)] = 0,$$

where we have used Lemma 3. Now, for $X_2 = \mathcal{P}X_2$, we get

$$cos^2\theta_2\, g(Y_2, X_2)[-(X_1 lnf) + \beta\eta(X_1)] = 0.$$

From the last expression, any one of the following can hold:

(i) if $cos^2\theta_2 = 0$, then $\theta_2 = \frac{\pi}{2}$ (i.e., $\mathcal{M}$ is a warped product pseudo-slant submanifold of $\overline{\mathcal{M}}$) or

(ii) if $\beta = 0$, then $f$ is constant (i.e., $\mathcal{M}$ is a Riemannian product) or

(iii) if $\beta \neq 0$, then $\beta\eta(X_1) = (X_1 lnf)$.

This completes the proof of our theorem.

For a warped product bi-slant submanifold in a trans-Sasakian manifold such that $\xi \in T\mathcal{M}_2$, we have the following result:

**Theorem 4.2:** *Let $\mathcal{M} = \mathcal{M}_1 \times_f \mathcal{M}_2$ be a warped product bi-slant submanifold with bi-slant angle $\{\theta_1, \theta_2\}$ in a trans-Sasakian manifold $\overline{\mathcal{M}}$ such that $\xi \in T\mathcal{M}_2$. Then one of the following these cases must occur:*

*(i)* $\mathcal{M}$ *is a warped product pseudo-slant submanifold such that $\mathcal{M}_2$ is a totally real submanifold $\mathcal{M}^\perp$ of $\overline{\mathcal{M}}$;*

*(ii)* $\mathcal{M}$ is a Riemannian product.

**Proof.** For any $X_1 \in T\mathcal{M}_1$ and $X_2, Y_2 \in T\mathcal{M}_2$, we have

$$\begin{aligned} g(h(X_1,X_2),\mathcal{F}Y_2) &= g(\overline{\nabla}_{X_2}X_1,\phi Y_2) - g(\nabla_{X_2}X_1,\mathcal{P}Y_2) \\ &= -(\mathcal{P}X_1 lnf)g(X_2,Y_2) + g(h(X_2,Y_2),\mathcal{F}X_1) \\ &\quad +(X_1 lnf)g(\mathcal{P}X_2,Y_2) \end{aligned} \tag{20}$$

Interchanging $X_2$ by $Y_2$ in equation (20), it follows that

$$\begin{aligned} g(h(X_1,Y_2),\mathcal{F}X_2) = &-(\mathcal{P}X_1 lnf)g(X_2,Y_2) + g(h(X_2,Y_2),\mathcal{F}X_1) \\ &+(X_1 lnf)g(\mathcal{P}Y_2,X_2). \end{aligned} \tag{21}$$

Subtracting (21) from (20), we obtain

$$g(h(X_1,X_2),\mathcal{F}Y_2) - g(h(X_1,Y_2),\mathcal{F}X_2) = 2(X_1 lnf)g(\mathcal{P}Y_2,X_2)$$

Using Lemma 1 and we deduce that

$$(X_1 lnf)g(\mathcal{P}Y_2,X_2) = 0$$

For $X_2 = \mathcal{P}X_2$, we get

$$cos^2\theta_2(X_1 lnf)[g(Y_2,X_2) - \eta(X_2)\eta(Y_2)] = 0.$$

Therefore, either

(i) $\theta_2 = \frac{\pi}{2}$ or

(ii) $f$ is constant.

Hence, our assertions follow.

We give an example of warped product bi-slant submanifold of the form $\mathcal{M} = \mathcal{M}_\theta \times_f \mathcal{M}_\perp$ whose bi-slant angles $\theta_1 \neq 0, \frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$. Such warped product bi-slant submanifolds are called pseudo-slant submanifolds. In particular, we can obtain an example of warped product pseudo-slant submanifold, of type $\mathcal{M}_\perp \times_f \mathcal{M}_\theta$ with $\xi \in T\mathcal{M}_\perp$, in the setting of Kenmotsu manifold. The example is as follows:

**Example 2:** Recalling example 3.1 in [13]. Consider the complex space $\mathbb{C}^4$ with the usual Kaehler structure and real global coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$. Let $\overline{\mathcal{M}} = \mathbb{R} \times_f \mathbb{C}^4$ be the warped product between the real line $\mathbb{R}$ and $\mathbb{C}^4$, where warping function is

$e^t$ and $t$ being the global coordinates in $\mathbb{R}$, then $\overline{\mathcal{M}}$ is a Kenmotsu manifold. Now defining the orthogonal basis

$$\mathcal{E}_1 = \partial/\partial x_1,$$
$$\mathcal{E}_2 = \partial/\partial y_3,$$
$$\mathcal{E}_3 = cos\theta\partial/\partial y_4 - sin\theta\partial/\partial x_4,$$
$$\mathcal{E}_4 = cos\theta\partial/\partial x_4 + sin\theta\partial/\partial y_4,$$
$$\mathcal{E}_5 = \partial/\partial t.$$

Obviously the distributions $\mathcal{D}_\theta = \{\mathcal{E}_3, \mathcal{E}_4\}$ and $\mathcal{D}_\perp = \{\mathcal{E}_5, \mathcal{E}_1, \mathcal{E}_2\}$ and denoted by $\mathcal{M}_\theta$ and $\mathcal{M}_\perp$, then $\mathcal{M} = \mathcal{M}_\theta \times_f \mathcal{M}_\perp$ is a pseudo-slant warped product submanifold isometrically immersed in $\overline{\mathcal{M}}$, here the warping function is $f = e^t$.

## 4. Doubly warped product bi-slant submanifolds

In general, doubly warped products can be considered as a generalization of warped products. Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be Riemannian manifolds. A doubly warped product $(\mathcal{M}, g)$ is a product manifold which is of the form $\mathcal{M} =_{f_2} \mathcal{M}_1 \times_{f_1} \mathcal{M}$ with the metric $g = f_1^2 g_1 \oplus f_2^2 g_2$, where $f_1 : \mathcal{M}_1 \times \mathcal{M}_2 \longrightarrow (0, \infty)$ and $f_2 : \mathcal{M}_1 \times \mathcal{M}_2 \longrightarrow (0, \infty)$ are smooth maps. More precisely, if $\rho : \mathcal{M}_1 \times \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ and $\delta : \mathcal{M}_1 \times \mathcal{M}_2 \longrightarrow \mathcal{M}_2$ are natural projections, the metric $g$ is defined by

$$g(X, Y) = (f_2 o \delta)^2 g_1(\rho^* X, \rho^* Y) + (f_1 o \rho)^2 g_2(\delta^* X, \delta^* Y) \tag{22}$$

for any $X, Y \in T\mathcal{M}$, where $*$ is the symbol for the tangent maps. The function $f_1$ and $f_2$ are called the warping functions of $\mathcal{M}$.

Remarks 5. If we assume

1. either $f_1 \equiv 1$ or $f_2 \equiv 1$, but not both, then we obtain a warped product.
2. both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we have a product manifold.
3. neither $f_1$ nor $f_2$ is constant, then we have a non-trivial doubly warped product.

Now, we define the notion of doubly warped product bi-slant submanifolds in a trans-Sasakian manifold as follows:

**Definition 6.** The doubly warped product of two slant submanifolds, ${}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ is called the doubly warped product bi-slant submanifold of slant submanifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ with slant angles $\theta_1$ and $\theta_2$, respectively, of a trans-Sasakian manifold with warping functions $f_1$ and $f_2$ if only depend on the points of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively.

For doubly warped product bi-slant submanifold $\mathcal{M}$ of a trans-Sasakian manifold $\overline{\mathcal{M}}$, we have

$$\nabla_Y X = \nabla_X Y = (Y ln f_1)X + (X ln f_2)Y \quad (23)$$

for any vector fields $X \in T\mathcal{M}_1$ and $Y \in T\mathcal{M}_2$.

**Theorem 4.3:** *Let $\mathcal{M} =_{f_2} \mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ be a doubly warped product submanifolds in a trans-Sasakian manifold $\overline{\mathcal{M}}$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are Riemannian submanifolds of $\overline{\mathcal{M}}$ and $\xi \in T\mathcal{M}_1$. Then $\mathcal{M}$ is a warped product submanifold in the form $\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ if and only if*

$$g(h(X,Y),\mathcal{F}X) = g(h(X,X),\mathcal{F}Y) \quad (24)$$

*for any vector fields $X \in T\mathcal{M}_1$ and $Y \in T\mathcal{M}_2$.*

**Proof.** From equation (2.12) of [19], we get

$$(\overline{\nabla}_X \mathcal{P})Y = A_{\mathcal{F}Y}X + \mathcal{B}h(X,Y)$$

for any vector fields $X \in T\mathcal{M}_1$ and $Y \in T\mathcal{M}_2$.

Applying (23) and we derive

$$(\mathcal{P}Y ln f_2)X - (Y ln f_2)\mathcal{P}X = \mathcal{B}h(X,Y) + A_{\mathcal{F}Y}X.$$

Taking inner product with $X \in T\mathcal{M}_1$, we obtain

$$(\mathcal{P}Y ln f_2)||X||^2 = g(h(X,X),\mathcal{F}Y) - g(h(X,Y),\mathcal{F}X).$$

Thus, from last relation, we conclude that $(\mathcal{P}Y ln f_2) = 0$ if and only if

$$g(h(X,Y),\mathcal{F}X) = g(h(X,X),\mathcal{F}Y)$$

for any vector fields $X \in T\mathcal{M}_1$ and $Y \in T\mathcal{M}_2$.

We conclude from $(\mathcal{P}Y ln f_2) = 0$ that $f_2$ depends only on the points of $\mathcal{M}_1$. Hence, $\mathcal{M}$ is a warped product bi-slant submanifold in the form $\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$. This completes the proof of the theorem.

At this moment, we need the following lemma which shall be helpful in proving next result of this paper:

**Lemma 2.** In a doubly warped product bi-slant submanifolds $\mathcal{M} =_{f_2} \mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ of a trans-Sasakian manifold $\overline{\mathcal{M}}$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are proper slant submanifolds with respect to $\theta_1$ and $\theta_2$, respectively and $\xi \in T\mathcal{M}_2$, the following relation holds

$$g(h(\mathcal{P}X,Z),\mathcal{F}X) = g(h(X,Z),\mathcal{F}\mathcal{P}X)$$

for any vector fields $X \in T\mathcal{M}_1$ and $Z \in T\mathcal{M}_2$.

**Proof.** For any vector fields $X \in T\mathcal{M}_1$ and $\xi, Z \in T\mathcal{M}_2$, we have

$$\begin{aligned} g(h(\mathcal{P}X,Z),\mathcal{F}X) &= g(\overline{\nabla}_Z \mathcal{P}X,\mathcal{F}X) \\ &= g(\mathcal{P}X,\overline{\nabla}_Z \mathcal{P}X) - g(\mathcal{P}X,\overline{\nabla}_Z \phi X) \end{aligned}$$

$$= g(\mathcal{P}X, \nabla_Z \mathcal{P}X) - g(\mathcal{P}X, (\overline{\nabla}_Z \phi)X) - g(\mathcal{P}X, \phi\overline{\nabla}_Z X$$
$$= cos^2\theta_1 (Zlnf_2)||X||^2 + g(\mathcal{F}\mathcal{P}X, h(X,Z) - cos^2\theta_1 g(X, \nabla_Z X)$$
$$= g(h(X,Z), \mathcal{F}\mathcal{P}X).$$

Hence, our assertion follows.

**Theorem 4.4:** *Let $\mathcal{M} =_{f_2} \mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ be a doubly warped product bi-slant submanifolds in a trans-Sasakian manifold $\overline{\mathcal{M}}$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are proper slant submanifolds with respect to $\theta_1$ and $\theta_2$, respectively and $\xi \in T\mathcal{M}_2$. Then $\mathcal{M}$ is a warped product bi-slant submanifold in the form $\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$.*

**Proof.** For any vector fields $X \in T\mathcal{M}_1$ and $Z, \xi \in T\mathcal{M}_2$, we have

$$g(h(\mathcal{P}X, X), \mathcal{F}Z) = g(\overline{\nabla}_{\mathcal{P}X} X, \phi Z)$$
$$= g((\overline{\nabla}_{\mathcal{P}X}\phi)X, Z) - g(\overline{\nabla}_{\mathcal{P}X}\phi X, Z)$$
$$= g(\nabla_{\mathcal{P}X} Z, \mathcal{P}X) + g(h(\mathcal{P}X, Z), \mathcal{F}X) - \beta cos^2\theta_1 \eta(Z)||X||^2$$
$$= cos^2\theta_1 (Zlnf_2)||X||^2 + g(h(\mathcal{P}X, Z), \mathcal{F}X)$$
$$-\beta cos^2\theta_1 \eta(Z)||X||^2. \quad (25)$$

On the other hand, using $\mathcal{P}X$ in the place of $X$, we derive

$$g(h(\mathcal{P}X, X), \mathcal{F}Z) = -cos^2\theta_1 (Zlnf_2)||X||^2 + g(h(X,Z), \mathcal{F}X)$$
$$-\beta cos^2\theta_1 \eta(Z)||X||^2. \quad (26)$$

Subtracting (25) from (26), we get

$$2cos^2\theta_1 (Zlnf_2)||X||^2 + g(h(\mathcal{P}X, Z), \mathcal{F}X) - g(\mathcal{F}\mathcal{P}X, h(X,Z)). \quad (27)$$

From (27) and Lemma 2, we find that $Zlnf_2 = 0$ for all $Z \in T\mathcal{M}_2$. This shows that $f_2$ depends only on the points of $\mathcal{M}_1$. Hence, $\mathcal{M}$ is a warped product bi-slant submanifold in the form $\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$. This proves the theorem completely.

## 5. Some Applications of the Theorem 3 for Different Kinds of Ambient Manifolds

**Corollary 1.** Let $\mathcal{M} =_{f_2} \mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ be a doubly warped product submanifolds in a Sasakian manifold $\overline{\mathcal{M}}$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are Riemannian submanifolds of $\overline{\mathcal{M}}$ and $\xi \in T\mathcal{M}_1$. Then $\mathcal{M}$ is a warped product submanifold in the form $\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ if and only if (24) holds.

**Corollary 2.** Let $\mathcal{M} =_{f_2} \mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ be a doubly warped product submanifolds in a Kenmotsu manifold $\overline{\mathcal{M}}$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are Riemannian submanifolds of $\overline{\mathcal{M}}$ and $\xi \in T\mathcal{M}_1$. Then $\mathcal{M}$ is a warped product submanifold in the form $\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ if and only if (24) holds.

**Corollary 3.** Let $\mathcal{M} ={}_{f_2} \mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ be a doubly warped product submanifolds in a cosymplectic manifold $\overline{\mathcal{M}}$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are Riemannian submanifolds of $\overline{\mathcal{M}}$ and $\xi \in T\mathcal{M}_1$. Then $\mathcal{M}$ is a warped product submanifold in the form $\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ if and only if (24) holds.

## 6. Some Applications of the Theorem 4 for Different Kinds of Ambient Manifolds

**Corollary 4.** Let $\mathcal{M} ={}_{f_2} \mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ be a doubly warped product bi-slant submanifolds in a Sasakian manifold $\overline{\mathcal{M}}$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are proper slant submanifolds with respect to $\theta_1$ and $\theta_2$, respectively and $\xi \in T\mathcal{M}_2$. Then $\mathcal{M}$ is a warped product bi-slant submanifold in the form $\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$.

**Corollary 5.** Let $\mathcal{M} ={}_{f_2} \mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ be a doubly warped product bi-slant submanifolds in a Kenmotsu manifold $\overline{\mathcal{M}}$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are proper slant submanifolds with respect to $\theta_1$ and $\theta_2$, respectively and $\xi \in T\mathcal{M}_2$. Then $\mathcal{M}$ is a warped product bi-slant submanifold in the form $\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$.

**Corollary 6.** Let $\mathcal{M} ={}_{f_2} \mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ be a doubly warped product bi-slant submanifolds in a cosymplectic manifold $\overline{\mathcal{M}}$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are proper slant submanifolds with respect to $\theta_1$ and $\theta_2$, respectively and $\xi \in T\mathcal{M}_2$. Then $\mathcal{M}$ is a warped product bi-slant submanifold in the form $\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$.

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# Quantum Information metric for time-dependent quantum systems and higher-order corrections

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**Abstract:** It is well established that quantum criticality is one of the most intriguing phenomena which signals the presence of new states of matter. Without prior knowledge of the local order parameter, the quantum information metric (or fidelity susceptibility) can indicate the presence of a phase transition as well as it measures distance between quantum states. In this work, we calculate the distance between quantum states which is equal to the fidelity susceptibility in quantum model for a time-dependent system describing a two-level atom coupled to a time-driven external field. As inspired by the Landau-Zener quantum model, we find in the present work information metric induced by fidelity susceptibility. We, for the first time, derive a higher-order rank-3 tensor as a third-order fidelity susceptibility. Having computed quantum noise function in this simple time-dependent model we show that the noise function eternally lasts long in our model.

**Keywords:** Fidelity Susceptibility; Quantum Information Theory; Information Metric 89.70.+c; 03.65.Ta; 52.65.Vv

## 1. Introduction

Quantum criticality is one of the most intriguing phenomena which is crucial for interpreting a wide variety of experiments. is well known, it signals the presence of new states of matter [1]. In order to observe exotic features at quantum critical point, one has to study systems in the thermodynamic regime involving large numbers of interacting particles, which encounter experimental and theoretical limitations [2]. Despite consisting only of a single-mode cavity field and a two-level atom, the authors of Ref.[3] show that the Rabi system exhibits a quantum phase transition (QPT). They demonstrate that the super radiant QPT primarily studied for systems of many atoms can be achieved with systems of a single one.
In recent years, there was a great deal of interest in studying QPTs from different perspectives of quantum information science [4], e.g., quantum entanglement [5, 6] and quantum fidelity [7, 8, 9, 10]. At the phase transition point, physical observables exhibit singular behavior governing the most dramatic manifestations of the laws of statistical and quantum mechanics.

In order to probe the phase transition, the fidelity susceptibility draws one of the most promising machines in which no prior knowledge of the order parameter and the symmetry of the system are required [13]-[20]. Regarding these works, the connection between the quantum information theory and condensed matter physics can be in principle achieved which might allow us to deepen our understanding in the various condensed matter phenomena. Notice that the concept of the fidelity susceptibility was originally introduced in Ref. [7]. In a recent study [21], the fidelity both in the susceptibility limit and the thermodynamic limit has been nicely summarize. Furthermore in Ref. [22], quantum information metric has been investigated near critical points.

An obvious physical example of QPTs using the quantum fidelity approach recently is given in [10],[22]. It was illustrated that at two sides of the critical point $g_c$ of a quantum many body system the ground state wavefunctions have different structures Ref. [11]. Then, consequently, this may lead to the overlap of the two ground states which are separated by a small distance $\delta g$ in the parameter space and then might emerge. In general, at the critical point $g_c$ the distance can be parameterized via $|\Psi_0(g)|\Psi_0(g+\delta g)|$ which is minimum. Therefore, the structure of the ground state of a quantum many-body system experiences a significant change because the system is driven across the transition point adiabatically. As a consequence, we expect that the fidelity susceptibility should be maximum (or even diverse) at the transition point, [7]. We notice that in various systems many authors have investigated the QPTs from the fidelity point of view [10, 7, 12]-[17]. They have shown that the fidelity susceptibility can be considered to be a simple approach in determining the universality of quantum phase transitions [12]-[17]. Interestingly, in [23], the quantum information metric gravity dual in conformal field theories has just been examined.

In this work, we study the fidelity susceptibility in quantum model for a time-dependent system describing a two-level atom coupled to a time-driven external field. We analytically investigate the behavior of fidelity susceptibility in the time driven quantum model when the potential $V$ is time-dependent. The organization of the paper is as follows. In Sec.2, we explore the mathematical foundations for fidelity susceptibility in time-dependent systems. In Sec.3, the two-level Landau-Zener problem is analyzed. In Sec.4, an experimental method based on noise function is proposed. In Sec.5, the higher-order correction to fidelity susceptibility is calculated. Finally, we conclude our findings in the last section.

## 2. Mathematical formulation of fidelity susceptibility in time-dependent driving systems

In this section we will formulate fidelity susceptibility for a general time-deriving system with two levels. Let us consider a physical system with non-perturbative time dependent Hamiltonian $H_0$ in operator form:

$$i\hbar\frac{\partial}{\partial t}\psi_k^{(0)} = H_0\psi_k^{(0)}. \tag{1}$$

Our aim is to find perturbed wavefunctions with Hamiltonian $H = H_0 + V(t)$ when $|V(t)| \ll |H_0|$. Note that here $V$ is considered to have off diagonal components, i.e, $V_{m\neq n} = \psi_m^{(0)}|V|\psi_n^{(0)} \neq 0$. Suppose that the perturbative solution for $H$ can be technically written in the following form:

$$\Psi = \sum_k a_k \psi_k^{(0)}, \tag{2}$$

where $a_k = a_k(t)$. Substituting (2) into Schrödinger equation and multiplying by $\psi_m^{(0)}$, we obtain:

$$i\hbar \frac{da_m}{dt} = \sum_k V_{mk}(t) a_k, \tag{3}$$

where

$$V_{mk}(t) = \int \psi_m^{*(0)} \hat{V} \psi_k^{(0)} dt = V_{mk} e^{i\frac{E_m^{(0)} - E_k^{(0)}}{\hbar} t}. \tag{4}$$

Using iteration method up to the first order, i.e. $a_k^{(0)} + a_k^{(1)}$ where $a_k^{(0)} = a_k(t = 0)$, we can find the ordinary differential equation for the first-order perturbation,

$$i\hbar \frac{da_k^{(1)}}{dt} = V_{kn}(t). \tag{5}$$

Finally, up to the first order perturbation theory, the total wave function is written as

$$\Psi_n = \sum_k a_{kn}(t) \psi_k^{(0)}. \tag{6}$$

Performing an integration, we obtain

$$a_{kn}^{(1)} = -\frac{i}{\hbar} \int V_{kn}(t) dt = -\frac{i}{\hbar} \int V_{kn} e^{i w_{kn} t} dt. \tag{7}$$

In this case, to figure out how $\chi_F$ looks like, we need the ground state wavefunction to be,

$$\psi_n = \psi_n^{(0)} + \sum_k a_{kn}^{(1)} \psi_k^{(0)}. \tag{8}$$

Let us further analyze our result for a two-level system. The perturbed wavefunction for the ground state $E_1$ is given by,

$$\psi_1^{(\lambda)} = (1 + \lambda_1 U_{11}) \psi_1^{(0)} + \lambda_2 W_{12} \psi_2^{(0)}. \tag{9}$$

Here we suppose that $a_{11}^{(1)} = \lambda_1 U_{11}, a_{12}^{(1)} = \lambda_2 W_{12}$. Let us calculate the inner product which is satisfied to yield the fidelity susceptibility, finally we suggest the following expression for the fidelity susceptibility $\chi_F$ for a time-driving system

$$\chi_{ij} = [\frac{\psi_1(\lambda)|\partial_{\lambda_i}\psi_1(\lambda)}{\psi_1|\psi_1}][\frac{\psi_1(\lambda)|\partial_{\lambda_j}\psi_1(\lambda)}{\psi_1|\psi_1}] + 2\frac{\psi_1(\lambda)|\partial_{\lambda_i}\partial_{\lambda_j}\psi_1(\lambda)}{\psi_1|\psi_1}\delta_{ij}. \quad (10)$$

Note that $d\hat{s}^2 = \chi_{ij}\delta\lambda_i\delta\lambda_j$ defines a Riemannian metric on a manifold $M$ which is a family of (positive definite) inner products – for all differentiable vector fields $\lambda_1, \lambda_2$ on $M$, that defines a smooth function $M \to R^2$ on coordinate space $(\lambda_i)^2$. An explicit form for the metric can be written as follows:

$$ds^2 = \chi_{11}d\lambda_1^2 + 2\mathrm{Re}(\chi_{12})d\lambda_1 d\lambda_2 + \chi_{22}d\lambda_2^2, \quad (11)$$

or its equivalent form,

$$ds^2 = \chi_{ij}(t)d\lambda_i d\lambda_j. \quad (12)$$

3.Fidelity susceptibility in the Landau-Zener problem

In the previous section we introduced a general formulation for fidelity susceptibility for time deriving potential. In this section, we will investigate a concrete example, inspired from Landau-Lifshitz cookbooks [29]. The system under consideration is a two-level quantum system initially prepared in ground state. The model named as Landau-Zener problem. The aim is to calculate $\chi_F$ matrix using (10). The ground state is defined by $n = 0$ and it satisfies:

$$H_{0_0} \quad \leq \quad H_{0_{\text{Excited state}}}. \quad (13)$$

The energy levels for the unperturbed Hamiltonian $H_0$ is defined as $E_a = E_1, E_2$ and it is convenient to define a frequency basis for the system,

$$\omega_{12} = \frac{E_2 - E_1}{\hbar} > 0. \quad (14)$$

As a two-level system, $E_1 = E_0 = E_{min}$, consequently we have:

$$E_2 > E_1. \quad (15)$$

The following two total wavefunctions of a two-level system $E_2 > E_1$ are defined using the orthogonality realization:

$$\Psi_1 = \sum_k a_{k1}(t)\psi_k^{(0)}, \quad \Psi_2 = \sum_k a_{k2}(t)\psi_k^{(0)}, \quad (16)$$

where

$$a_{kn} = \delta_{kn} - \frac{i}{\hbar}\int V_{kn} e^{i\omega_{kn}t} dt. \tag{17}$$

Next, we propose a specific form of the potential as

$$V = Fe^{-i\omega t} + Ge^{i\omega t}, \tag{18}$$

where $F$ and $G$ are time-independent operators. If $V_{nm} = V_{mn}^*$ then we obtain $G_{nm} = F_{mn}^*$. In this situation, the matrix element takes the form,

$$V_{kn}(t) = V_{kn} e^{i\omega_{kn}t} = F_{kn} e^{i(w_{kn}-\omega)t} + F_{kn}^* e^{i(w_{kn}+\omega)t}. \tag{19}$$

Substituting (19) into (17) and performing an integration, we obtain

$$a_{kn}^{(1)} = -\frac{F_{kn} e^{i(\omega_{kn}-w)t}}{\hbar(\omega_{kn}-\omega)} - \frac{F_{kn}^* e^{i(\omega_{kn}+w)t}}{\hbar(\omega_{kn}+\omega)}, \tag{20}$$

where we have assumed that $\omega_{kn} \neq \pm\omega$. Note that the matrix element for an arbitrary operator $O$ is given by:

$$O_{mn}(t) = O_{mn}^{(0)} e^{i\omega_{nmt}} + O_{mn}^{(1)}(t), \tag{21}$$

where

$$O_{mn}^{(1)}(t) = e^{i\omega_{nm}t}\Big(\sum_k \Big[\frac{O_{nk}^{(0)} F_{km}}{\hbar(\omega_{km}-\omega)} + \frac{O_{km}^{(0)} F_{nk}}{\hbar(\omega_{kn}+\omega)}\Big] e^{-i\omega t} + \Big[\frac{O_{nk}^{(0)} F_{mk}^*}{\hbar(\omega_{mk}+\omega)} + \frac{O_{km}^{(0)} F_{kn}^*}{\hbar(\omega_{nk}-\omega)}\Big] e^{i\omega t}\Big). \tag{22}$$

To be more concrete when choosing $O = H$ and $H_{nk}^{(0)} = E_k \delta_{nk}$, the matrix form for $H$ in zeroth order reads,

$$H_{nm} = E_m \delta_{nm} e^{i\omega_{nm}t} - e^{i\omega_{nm}t}\Big(\sum_k \Big[\frac{E_k \delta_{nk} F_{km}}{\hbar(\omega_{km}-\omega)} + \frac{E_k \delta_{mk} F_{nk}}{\hbar(\omega_{kn}+\omega)}\Big] e^{-i\omega t} + \Big[\frac{E_k \delta_{nk} F_{mk}^*}{\hbar(\omega_{mk}+\omega)} + \frac{E_k \delta_{mk} F_{kn}^*}{\hbar(\omega_{nk}-\omega)}\Big] e^{i\omega t}\Big). \tag{23}$$

If $F$ is real, i.e., $F_{mn} = F_{nm}^*$, we obtain the following expression for a matrix representation of $H$ up to the first-order perturbation,

$$H_{nm} = E_n\delta_{nm}e^{i\omega_{nm}t} - e^{i\omega_{nm}t}F_{nm}\omega_{nm}\left(\frac{e^{-i\omega t}}{\omega_{nm}-\omega} + \frac{e^{i\omega t}}{\omega_{nm}+\omega}\right). \tag{24}$$

Note that the diagonal elements are commonly parametrized by $H_{nn} = E_n$ and the off-diagonal ones are

$$H_{n\neq m} = -e^{i\omega_{nm}t}F_{nm}\omega_{nm}\left(\frac{e^{-i\omega t}}{\omega_{nm}-\omega} + \frac{e^{i\omega t}}{\omega_{nm}+\omega}\right). \tag{25}$$

For the two-level system, it is still plausible to obtain

$$H_{12} = (H_{21})^* = \omega_0 F_{12}\left(\frac{e^{i(\omega-\omega_0)t}}{\omega-\omega_0} - \frac{e^{-i(\omega+\omega_0)t}}{\omega+\omega_0}\right).$$

The wavefunction coefficients read as follows:

$$a_{11}^{(1)} = i\frac{F_{11}}{\hbar\omega}\sin(\omega t), \tag{26}$$

and

$$a_{21}^{(1)} = -\frac{F_{12}^*}{\hbar}\left[\frac{e^{-i\omega t}}{\omega_0-\omega} + \frac{e^{i\omega t}}{\omega_0+\omega}\right]. \tag{27}$$

Therefore, the total perturbed wavefunction for the ground state is given by,

$$\Psi_1 = \left(\frac{F_{11}}{\hbar\omega}\sin(\omega t)\right)\psi_1^{(0)} - \frac{F_{12}^*}{\hbar}\left(\frac{e^{-i\omega t}}{\omega_0-\omega} + \frac{e^{i\omega t}}{\omega_0+\omega}\right)\psi_2^{(0)}. \tag{28}$$

It is reasonable to parametrize perturbed matrix elements as follows:

$$F_{11} = \lambda_1 V_{11}, \tag{29}$$

$$F_{12} = \lambda_2 W_{12}. \tag{30}$$

In terms of these parameters, we obtain

$$\Psi_1 = \left(\frac{iV_{11}}{\hbar\omega}\sin\omega t\right)\lambda_1\psi_1^{(0)} - \frac{iW_{12}^*}{\hbar}e^{i\omega_0 t}\left(\frac{e^{-i\omega t}}{\omega_0-\omega} + \frac{e^{i\omega t}}{\omega_0+\omega}\right)\lambda_2\psi_2^{(0)}. \tag{31}$$

By defining two auxiliary functions,

$$\alpha(t) = \frac{iV_{11}}{\hbar\omega}\sin\omega t, \tag{32}$$

$$\beta(t) = \frac{-iW_{12}^{*}}{\hbar}\left(\frac{e^{i(\omega_0-\omega)t}}{\omega_0-\omega}+\frac{e^{i(\omega_0+\omega)t}}{\omega_0+\omega}\right), \tag{33}$$

and using (10), we end up with the matrix elements for $\chi_F$ as follows:

$$\chi_{11} = \frac{1}{2\lambda_1}\frac{1}{1+|\frac{\beta}{\alpha}|^2\left(\frac{\lambda_2}{\lambda_1}\right)^2}, \tag{34}$$

$$\chi_{12} = \frac{1}{2\lambda_1}\frac{1+|\frac{\beta}{\alpha}|^2\left(\frac{\lambda_2}{\lambda_1}\right)}{1+|\frac{\beta}{\alpha}|^2\left(\frac{\lambda_2}{\lambda_1}\right)^2}, \tag{35}$$

$$\chi_{22} = \frac{1}{2\lambda_1}\frac{|\frac{\beta}{\alpha}|^2}{1+|\frac{\beta}{\alpha}|^2(\frac{\lambda_2}{\lambda_1})^2}. \tag{36}$$

Note that here $\lambda_2 \neq \lambda_1$ to have the non-singular metric $\chi_{ij}$. In our model,

$$|\frac{\beta}{\alpha}|^2 = |\frac{2\omega W_{12}^{*}}{V_{11}}|^2(\frac{\omega^2\cos^2(\omega_0 t)+\omega_0^2\sin^2(\omega_0 t)\cot^2(\omega t)}{(\omega^2-\omega_0^2)^2})$$

We are interested in high frequencies where $\omega \gg \omega_0$. In this case we have

$$|\frac{\beta}{\alpha}|^2 \approx |\frac{4W_{12}^{*}}{V_{11}}|^2\cos^2(\omega_0 t). \tag{37}$$

Finally, by defining $\gamma \equiv |\frac{4W_{12}^{*}}{V_{11}}|^2 > 0$, we have the following approximated form for fidelity susceptibility at high frequencies and ultraviolet (UV) regime as follows:

$$\chi_{11} = \frac{1}{2\lambda_1}\frac{1}{1+\gamma\cos^2(\omega_0 t)(\frac{\lambda_2}{\lambda_1})^2}, \tag{38}$$

$$\chi_{12} = \frac{1}{2\lambda_1}\frac{1+\gamma\cos^2(\omega_0 t)(\frac{\lambda_2}{\lambda_1})}{1+\gamma\cos^2(\omega_0 t)(\frac{\lambda_2}{\lambda_1})^2}, \tag{39}$$

$$\chi_{22} = \frac{1}{2\lambda_1}\frac{\gamma\cos^2(\omega_0 t)}{1+\gamma\cos^2(\omega_0 t)(\frac{\lambda_2}{\lambda_1})^2}. \tag{40}$$

The information metric, measures the distance between two quantum states close to each other in UV regime and is given as follows:

$$ds^2 = \frac{1}{2\lambda_1(1+\gamma\cos^2(\omega_0 t)(\frac{\lambda_2}{\lambda_1})^2)}[d\lambda_1^2 + 2(1+\gamma\cos^2(\omega_0 t)(\frac{\lambda_2}{\lambda_1}))d\lambda_1 d\lambda_2 + \gamma\cos^2(\omega_0 t)d\lambda_2^2]. \quad (41)$$

This metric could be dual to a non-relativistic time dependent bulk theory via Maldacena's AdS/CFT correspondence [37] in a same methodology as presented in [38].

4.Measurement $\boldsymbol{\chi_F}$ using quantum noise setups

In recent years, the time-dependent systems phase transitions have been investigated in references [30]-[36]. In Ref. [30], the universal scaling behavior in a one-dimensional quantum Ising model subject to time-dependent sinusoidal modulation in time of its transverse magnetic field has been illustrated. This scaling behavior existed in various quantities, e.g. concurrence, entanglement entropy, magnetic and fidelity susceptibility. Based on an Ising spin chain and with periodically varying external magnetic field along the transverse direction the authors, in Ref. [31], investigated the microscopic quantum correlations dynamics of the bipartite entanglement and quantum discord.

In this section, we mainly focus on frequency spectrum of the quantum system resulting from the quantum noise function. Let us assume a generalized Hamiltonian $H = H_0 + \lambda V$ with $\lambda$ denoting the control parameter. The quantum noise spectrum of the driven Hamiltonian $V$ can be defined as

$$S_Q(\omega) = \sum_{n\neq 0} |\langle\phi_n|V|\phi_0\rangle|^2 \delta(\omega - E_n + E_0), \quad (42)$$

where $\phi_n$ is the eigenstate of the Hamiltonian $H(\lambda)$ and we assumed $E_n$ as non-degenerate energy levels of the whole system. Note that the quantum noise function $S_Q(\omega)$ can be constructed from the excited states $E_n > E_0$. In our model, the ground state wavefunction is given in Eq. (31). Here we can rewrite the noise function (42) using matrix element given in Eq. (19) as follows:

$$S_Q(\omega) == 2|\lambda_2 W_{12}|^2 \cos^2 \omega t. \quad (43)$$

We plot the noise function $S_Q(\omega)$ versus time $(t)$ and frequency $(\omega)$ illustrated in Fig.(1).

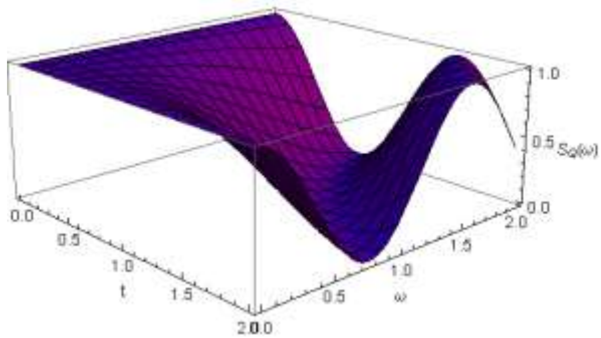


Figure 1: The plot shows the noise function $S_Q(\omega)$ as functions of time $(t)$ and frequency $(\omega)$.

As well known, the fidelity susceptibility plays an important role in QPTs stemming from the fact that it is always possible to describe the universality classes of QPTs without specifying the type of the symmetry of the system. However, it is adequate to ask whether we can measure $\chi_F$ using experimental setups. It has been shown that recently the $\xi_F$ is related to the quantum noise spectrum of the time-driven Hamiltonian [27]. It is remarkable to relate $\xi_F$ to $S_Q(\omega)$ using Kronig-Penney transformation:

$$\chi_F = \int_{-\infty}^{\infty} d\omega \frac{S_Q(\omega)}{\omega^2}. \tag{44}$$

Bearing in mind that the following definition of derivative of any analytic function $f(z)$ provides a useful tool:

$$f(z): \mathcal{C} \to \mathcal{C},$$
$$f^{(n)}(z) = \frac{\Gamma(n+1)}{2\pi i} \int \frac{f(w)}{(w-z)^{n+1}} dw, \tag{45}$$

where $\Gamma(n+1) = \int_0^{\infty} e^{-t} t^n dt$ is a Gamma function. Using (45) we clearly observe that [27]:

$$\chi_F = \pi i \frac{d^2 S_Q(\omega)}{d\omega^2}|_{\omega=0}. \tag{46}$$

It is clearly stated that $S_Q(\omega)$ can be measured in laboratory, see Ref. [28]. Consequently, we verify that the $\chi_f$ could be measured in the laboratory, as well. Particularly the Landau-Lifshitz model with $\chi_F$ presented in Eqs. (54)-(56) provides a useful machinery to study the universal scaling behavior of $\chi_F$.

## 5.$\mathcal{O}(\delta\lambda)^3$ Missing term

In this section, we highlight higher order corrections up to the $\mathcal{O}(\delta\lambda^3)$ of $\chi_F$. It is noteworthy to figure out higher order terms, i.e., the coefficient of $\delta\lambda^2$ using the expressions given above. Remember that

$$\psi(\lambda+\delta\lambda)=\psi(\lambda)+\delta\lambda\partial_\lambda\psi(\lambda)+\frac{\delta\lambda^2}{2}\partial_\lambda^2\psi(\lambda). \quad (47)$$

Let us compute the following inner product:

$$\psi(\lambda)|\psi(\lambda+\delta\lambda)\approx\psi(\lambda)|\psi(\lambda)+\delta\lambda\psi(\lambda)|\partial_\lambda\psi(\lambda)+\delta\lambda^2\psi(\lambda)|\partial_\lambda^2\psi(\lambda)+\ldots, \quad (48)$$

where the ellipses denote higher order (correction) terms. Consequently, we obtain the following expression for the third-order fidelity susceptibility as follows:

$$\zeta_F=\frac{\psi(\lambda)|\partial_\lambda\psi(\lambda)\psi(\lambda)|\partial_\lambda^2\psi(\lambda)}{|\psi|\psi|^2}. \quad (49)$$

The above equation defines a higher order correction to the usual fidelity susceptibility. The corresponding metric is a Finsler manifold in which the general information metric is characterized by the following form:

$$ds^2=\chi_{ij}d\lambda_i d\lambda_j+(\zeta_{ijk}d\lambda_i d\lambda_j\lambda_k)^{\frac{2}{3}}+\ldots. \quad (50)$$

It is worth noting that the distant between two quantum states in any quantum theory can be quantified not only by fidelity but also with higher order cubic quantity defined by $\zeta_F$. We note here that the corresponding tensor form for $\zeta_F$ is given by:

$$(\zeta_F)_{ijk}=\frac{\psi(\lambda)|\partial_{\lambda_i}\psi(\lambda)\psi(\lambda)|\partial_{\lambda_j}\partial_{\lambda_k}\psi(\lambda)}{|\psi|\psi|^2}. \quad (51)$$

It will be very interesting to find bulk dual for this new tensor in a similar way recently suggested for fidelity susceptibility as a maximal volume in the AdS spacetime [38].

## 6.Summary

In this work, we have presented a simple and straightforward approach to compute distance between quantum states responsible for the fidelity susceptibility in quantum model for a time-dependent system describing a two-level atom coupled to a time-driven external field. Analytically we have investigated the behavior of fidelity susceptibility in the time-driven quantum model in which the potential $V$ is time-dependent. Interestingly, the information metric induced by fidelity susceptibility can be nicely achieved. We also plotted the obtained noise function and found that the noise function eternally lasts long in our model. We have also derived for the first time a higher-order rank-3 tensor as third-order fidelity susceptibility for having a model beyond fidelity susceptibility.

It will be very interesting to find bulk dual for this new tensor in a similar way recently suggested for fidelity susceptibility as a maximal volume in the AdS spacetime [38]. Moreover, as mentioned in Refs. [39, 40], our understanding of quantum gravity may be satisfied using quantum information theory along with holography. This may allow us to further examine a possible connection between the fidelity susceptibility and holographic complexity and may shed new light on the deeper understanding of quantum gravity.

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# Weak convergence to common attractive points of finite families of nonexpansive mappings

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**Abstract:** In this paper, we introduce an iterative method to approximate a common solution of attractive fixed point problems for a finite family of nonlinear mappings in a real Hilbert space using the Ces`$a$ro mean approximation method. We obtain a weak convergence theorem for the sequences generated by the proposed iterative scheme. Finally, we derive some consequences from our main result. The results presented in this paper extended and unify many of the previously known results in this area.

**Keywords**: Nonexpansive mapping; Common attractive fixed point problem; Ces`$a$ro mean approximation method.
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## 1.Introduction

Throughout the paper unless otherwise stated, let $H$ be a real Hilbert space with inner product $\langle .,. \rangle$ and $\|.\|$ denotes the norm of $H$. Let $C$ be a nonempty subset of $H$. A mapping $T: C \to H$ is said to be a generalized hybrid mapping [1] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta \| Tx - y \|^2 + (1-\beta) \| x - y \|^2, \forall\, x, y \in C.$$

A mapping $T: C \to H$ is said to be a nonexpansive if $\alpha = 1$ and $\beta = 0$ that is

$$\| Tx - Ty \| \leq \| x - y \|, \forall\, x, y \in C.$$

A mapping $T: C \to H$ is said to be a nonspreading [2] if $\alpha = 2$ and $\beta = 1$ that is

$$2 \| Tx - Ty \|^2 \leq \| Tx - y \|^2 + \| Ty - x \|^2, \forall x, y \in C.$$

It is said to be a hybrid [3] if $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$ that is

$$3 \| Tx - Ty \|^2 \leq \| x - y \|^2 + \| Tx - y \|^2 + \| Ty - x \|^2, \forall x, y \in C.$$

Takahashi and Takeuchi [4] introduced the concept of attractive points of a nonlinear mapping in a Hilbert space and they proved a mean convergence theorem of Baillon's type

[5] without convexity for generalized hybrid mappings.

Recently, Takahashi *et al*. [6] studied the following Halpern's type [7] iterative scheme and proved a strong convergence theorem for finding attractive points of generalized hybrid mappings in a Hilbert space

$$x_{n+1} = \alpha_n z + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n).$$

Very recently, Zheng [8] studied the following Ishikawa iterative scheme and proved weak and strong convergence theorems for finding attractive points of generalized hybrid mappings in Banach space

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n. \end{cases}$$

Motivated by the work of Takahashi and Takeuchi [4], Takahashi et al. [6], Zheng [8], and by the ongoing research in this direction, we introduce an iterative method based on viscosity implicit midpoint method for finding an attractive point in a real Hilbert space. We obtain a weak convergence theorem for the sequence generated by the proposed iterative scheme. Finally, we derive some consequence from our main result. The result presented in this paper extended and unify many of the previously known results in this area, see instance [1, 2, 5, 9].

## 2.Preliminaries

We recall some concepts and results that are needed in the sequel.
The symbols $\rightarrow$ and $\rightharpoonup$ denote strong and weak convergence, respectively, $I$ denotes the identity operator on $H$.

For every point $x \in H$, there exists a unique nearest point to $x$ in $C$ denoted by $P_C x$ such that

$$\| x - P_C x \| \leq \| x - y \|, \forall y \in C. \tag{2.1}$$

The mapping $P_C$ is called the metric projection of $H$ onto $C$. It is well known that $P_C$ is nonexpansive and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \| P_C x - P_C y \|^2, \forall x, y \in H. \tag{2.2}$$

Moreover, $P_C x$ is characterized by the fact that $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \geq 0, \forall y \in C. \tag{2.3}$$

This implies that

$$\| x - y \|^2 \geq \| x - P_C x \|^2 + \| y - P_C x \|^2, \forall x \in H, \forall y \in C. \tag{2.4}$$

In real Hilbert space $H$, it is well known that

$$\| \lambda x + (1-\lambda) y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda(1-\lambda) \| x - y \|^2, \tag{2.5}$$

for all $x, y \in H$ and $\lambda \in [0,1]$ and

$$\| x + y \|^2 \leq \| x \|^2 + 2\langle y, x + y\rangle, \forall x, y \in H. \tag{2.6}$$

Let $T: C \to H$ be a mapping. The set of attractive points of $T$ denoted by $A(T)$ and defined as

$$A(T) = \{y \in H: \| Tx - y \| \leq \| x - y \|, \forall x \in C\}. \tag{2.7}$$

**Lemma 1.1** *[4] Let $C$ be a nonempty subset $H$ and let $T$ be a mapping from $C$ into $H$. Then, $A(T)$ is closed and convex subset of $H$.*

**Lemma 1.2** *[4] Let $C$ be a nonempty subset of $H$ and let $T: C \to H$ be a generalized hybrid mapping from $C$ into self. Suppose that there exists an $x \in C$ such that $\{T^n x\}$ is bounded. Then, $A(T) \neq \phi$.*

**Lemma 1.3** *[4] Let $C$ be a nonempty subset of $H$. Let $T: C \to H$ be a quasi-nonexpansive mapping. Then, $A(T) \cap C = Fix(T)$.*

**Lemma 1.4** *[6] Let $C$ be a nonempty subset of $H$. Let $T: C \to H$ be a generalized hybrid mapping. If $x_n \rightharpoonup x_0$ and $x_n - Tx_n \to 0$, then $x_0 \in A(T)$.*

## 3. Main results

We prove a weak convergence theorem for finding attractive points of a generalized hybrid mapping in a Hilbert space.

**Theorem 4.5:** *Let $H$ be a real Hilbert space and let $C$ be a nonempty convex subset of a real Hilbert space $H$. Let $T_i: C \to C$ be nonexpansive mappings for each $i = 0,1,2,\ldots,m-1$ with $A(T_i) \neq \emptyset$ and let $P_{\bigcap_{i=0}^{m-1} A(T_i)}$ be the metric projection of $H$ onto $\bigcap_{i=0}^{m-1} A(T_i)$. Let $\{x_n\}$ be a sequence in $C$ generated by*

$$\begin{aligned} x_1 &\in C \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n) y_n \\ y_n &= \beta_n x_n + (1-\beta_n)\frac{1}{m}\sum_{i=0}^{m-1} T_i x_n, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that $\lim\, inf_{n\to\infty}\beta_n(1-\alpha_n)(1-\beta_n) > 0$. Then the sequence $\{x_n\}$ converges weakly to $\bar{x} = P_{\cap_{i=0}^{m-1} A(T_i)} z$.

**Proof.** First, we show that the sequence $\{x_n\}$ is bounded. Let $p \in \cap_{i=0}^{m-1} A(T_i)$.
We estimate

$$\begin{aligned} &\| x_{n+1} - p \| = \| \alpha_n x_n + (1-\alpha_n) y_n - p \| \\ &= \| \alpha_n (x_n - p) + (1-\alpha_n)(y_n - p) \| \\ &\le \alpha_n \| x_n - p \| + (1-\alpha_n) \| y_n - p \| \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\| y_n - p \| \le \| \beta_n x_n + (1-\beta_n)\frac{1}{m}\sum_{i=0}^{m-1} T_i x_n - p \| \\ &= \| \beta_n (x_n - p) + (1-\beta_n)\left(\frac{1}{m}\sum_{i=0}^{m-1} T_i x_n - p\right) \| \\ &\le \beta_n \| x_n - p \| + (1-\beta_n) \| \frac{1}{m}\sum_{i=0}^{m-1} T_i x_n - p \| \\ &\le \beta_n \| x_n - p \| + (1-\beta_n) \| x_n - p \| \\ &= \| x_n - p \|. \end{aligned} \tag{3.2}$$

Using (3.2) in (3.1)

$$\begin{aligned} \| x_{n+1} - p \| &\le \alpha_n \| x_n - p \| + (1-\alpha_n) \| x_n - p \| \\ &= \| x_n - p \| \\ &\quad . \\ &\quad . \\ &\quad . \\ &\le \| x_1 - p \|. \end{aligned}$$

Hence, $\{x_n\}$ is bounded. Since $p \in \cap_{i=0}^{m-1} A(T_i)$ therefore $\| T_i x_n - p \| \le \| x_n - p \|$, for each $i = 0,1,2,\dots,m-1$. Thus, $\{T_i x_n\}$ is bounded and hence $\{y_n\}$ is also bounded.

Using (2.5), we estimate

$$\| x_{n+1} - p \|^2 = \| \alpha_n x_n + (1-\alpha_n) y_n - p \|^2 \tag{3.3}$$

$$=\| \alpha_n(x_n-p)+(1-\alpha_n)(y_n-p) \|^2$$
$$\leq \alpha_n \| x_n-p \|^2+(1-\alpha_n) \| y_n-p \|^2-\alpha_n(1-\alpha_n) \| x_n-y_n \|^2$$
$$\leq \alpha_n \| x_n-p \|^2+(1-\alpha_n) \| y_n-p \|^2,$$

and

$$\| y_n-p \|^2=\| \beta_n x_n+(1-\beta_n)\frac{1}{m}\sum_{i=0}^{m-1} T_i x_n-p \|^2$$
$$=\| \beta_n(x_n-p)+(1-\beta_n)(\frac{1}{m}\sum_{i=0}^{m-1} T_i x_n-p) \|^2$$
$$\leq \beta_n \| x_n-p \|^2+(1-\beta_n) \| \frac{1}{m}\sum_{i=0}^{m-1} T_i x_n-p \|^2$$
$$-\beta_n(1-\beta_n) \| \frac{1}{m}\sum_{i=0}^{m-1} T_i x_n-x_n \|^2. \tag{3.4}$$

Using (3.4) in (3.3), we have

$$\| x_{n+1}-p \|^2= \alpha_n \| x_n-p \|^2+(1-\alpha_n)\beta_n \| x_n-p \|^2$$
$$+(1-\alpha_n)(1-\beta_n) \| \frac{1}{m}\sum_{i=0}^{m-1} T_i x_n-p \|^2$$
$$-\beta_n(1-\alpha_n)(1-\beta_n) \| \frac{1}{m}\sum_{i=0}^{m-1} T_i x_n-p \|^2.$$

This implies that

$$\beta_n(1-\alpha_n)(1-\beta_n)\frac{1}{m}\sum_{i=0}^{m-1} T_i x_n-p \|^2\leq \alpha_n \| x_n-p \|^2+(1-\alpha_n)\beta_n$$
$$\| x_n-p \|^2$$
$$+(1-\alpha_n)(1-\beta_n) \| \tfrac{1}{m}\textstyle\sum_{i=0}^{m-1} T_i x_n-p \|^2 -\| x_{n+1}-p \|^2$$
$$\leq \alpha_n \| x_n-p \|^2+(1-\alpha_n)\beta_n \| x_n-p \|^2$$
$$+(1-\alpha_n)(1-\beta_n) \| x_n-p) \|^2 -\| x_{n+1}-p \|^2$$
$$\leq \alpha_n \| x_n-p \|^2+(1-\alpha_n) \| x_n-p \|^2 -\| x_{n+1}-p \|^2$$
$$=\| x_n-p \|^2 -\| x_{n+1}-p \|^2. \tag{3.5}$$

Now, summing

$$\sum_{n=1}^{\infty} \beta_n(1-\alpha_n)(1-\beta_n) \parallel \frac{1}{m}\sum_{i=0}^{m-1} T_i x_n - x_n \parallel^2 \leq \parallel x_1 - p \parallel^2 < \infty$$

Using the given condition, we have

$$\lim_{n\to\infty} \parallel t_m x_n - x_n \parallel = 0, \text{where } t_m = \frac{1}{m}\sum_{i=0}^{m-1} T_i. \tag{3.6}$$

$$\begin{aligned}\parallel x_{n+1} - x_n \parallel &= \parallel \alpha_n x_n + (1-\alpha_n) y_n - x_n \parallel \\ &= \parallel (1-\alpha_n)(y_n - x_n) \parallel \\ &= \parallel (1-\alpha_n)(\beta_n x_n + (1-\beta_n) t_m x_n - x_n) \parallel \\ &= (1-\alpha_n)(1-\beta_n) \parallel x_n - t_m x_n \parallel. \end{aligned} \tag{3.7}$$

Using (3.6) in (3.7) and given condition, we have

$$\lim_{n\to\infty} \parallel x_{n+1} - x_n \parallel = 0. \tag{3.8}$$

Since $\{x_n\}$ is bounded therefore there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. By (3.6) and Lemma 2.4, we have that $w \in A(t_m)$. Thus, for each $i = 0, 1, 2, \ldots, m-1$,

$$w \in \bigcap_{i=0}^{m-1} A(T_i).$$

Finally, we prove that $\{x_n\}$ converges weakly to $w$. On contrary suppose that there exists another subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which weakly converges to some $w' \neq w$. By (3.6) and Lemma 2.4, we have that $w' \in A(t_m)$. Thus, for each $i = 0,1,2,\ldots, m-1$,

$$w' \in \bigcap_{i=0}^{m-1} A(T_i).$$

Now,

$$\begin{aligned}\lim_{n\to\infty} \parallel x_n - w \parallel^2 &= \lim_{k\to\infty} \parallel x_{n_k} - w \parallel^2 \\ &= \lim_{k\to\infty} (\parallel x_{n_k} - w' \parallel^2 + 2\langle x_{n_k} - w', w' - w\rangle + \parallel w' - w \parallel^2) \\ &= \lim_{k\to\infty} \parallel x_{n_k} - w' \parallel^2 + 2\langle w - w', w' - w\rangle + \parallel w' - w \parallel^2 \\ &= \lim_{k\to\infty} \parallel x_{n_k} - w' \parallel^2 - \parallel w' - w \parallel^2 \\ &= \lim_{n\to\infty} \parallel x_n - w' \parallel^2 - \parallel w' - w \parallel^2 \\ &= \lim_{i\to\infty} \parallel x_{n_i} - w' \parallel^2 - \parallel w' - w \parallel^2 \end{aligned}$$

$$\begin{aligned}
&= \lim_{i\to\infty} \| x_{n_i} - w' \|^2 - \| w' - w \|^2 \\
&= \lim_{i\to\infty} (\| x_{n_i} - w' \|^2 + 2\langle x_{n_i} - w, w - w' \rangle + \| w - w' \|^2) \\
&\quad - \| w' - w \|^2 \\
&= \lim_{i\to\infty} \| x_{n_i} - w \|^2 - 2 \| w - w' \|^2 \\
&= \lim_{n\to\infty} \| x_n - w \|^2 - 2 \| w - w' \|^2.
\end{aligned}$$

This implies that $w = w'$, a contradiction.
Thus, the sequence $\{x_n\}$ converges weakly to an attractive point $w$ of $T_i$, for each $i = 0,1,2,\ldots,m-1$.

Using Theorem 3.1, we can prove the following convergence theorems:

**Theorem 4.6:** *Let $H$ be a real Hilbert space and let $C$ be a nonempty closed and convex subset of $H$. Let $T_i: C \to C$ be a generalized hybrid mapping for each $i = 0,1,2,\ldots,m-1$ with $A(T_i) \neq \emptyset$ and let $P_{\cap_{i=0}^{m-1} Fix(T_i)}$ be the metric projection of $H$ onto $\cap_{i=0}^{m-1} Fix(T_i)$. Let $\{x_n\}$ be a sequence in $C$ generated by*

$$\begin{aligned}
x_1 &\in C \\
x_{n+1} &= \alpha_n x_n + (1-\alpha_n) y_n \\
y_n &= \beta_n x_n + (1-\beta_n) \frac{1}{m} \sum_{i=0}^{m-1} T_i x_n,
\end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that $liminf_{n\to\infty} \beta_n (1-\alpha_n)(1-\beta_n) > 0$. Then the sequence $\{x_n\}$ converges weakly to

$$\bar{x} = P_{\cap_{i=0}^{m-1} Fix(T_i)} w.$$

**Proof.** For each $i = 0,1,2,\ldots,m-1$, $T_i: C \to C$ be a generalized hybrid mapping then there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \| T_i x - T_i y \|^2 + (1-\alpha) \| x - T_i y \|^2 \leq \beta \| T_i x - y \|^2 + (1-\beta) \| x - y \|^2, \forall x, y \in C. \tag{3.9}$$

Let $w \in \cap_{i=0}^{m-1} \mathrm{F}ix(T_i)$ and replacing $x$ by $w$ in (3.9), we have

$$\| T_i y - w \| \leq \| y - w \|, \forall y \in C.$$

This implies that

$$w \in \bigcap_{i=0}^{m-1} \mathrm{F}ix(T_i).$$

Thus,

$$\bigcap_{i=0}^{m-1} \mathrm{F}ix(T_i) \subset \bigcap_{i=0}^{m-1} A(T_i).$$

Hence,

$$\bigcap_{i=0}^{m-1} A(T_i) \neq \emptyset.$$

By Theorem 3.1, it follows that $\{x_n\}$ converges weakly to

$$\bar{x} \in \bigcap_{i=0}^{m-1} A(T_i).$$

Since $C$ is closed and convex therefore by Lemma 3.3,

$$\bigcap_{i=0}^{m-1} A(T_i) \bigcap \ C = \bigcap_{i=0}^{m-1} \mathrm{F}ix(T_i).$$

Thus, $\{x_n\}$ converges weakly to an element

$$\in \bigcap_{i=0}^{m-1} \mathrm{F}ix(T_i).$$

**Theorem 4.7:** *Let $H$ be a real Hilbert space and let $C$ be a nonempty convex subset of $H$. Let $T_i: C \to C$ be a nonexpansive mapping for each $i = 0,1,2,\ldots,m-1$ with $A(T_i) \neq \emptyset$ and let $P_{\bigcap_{i=0}^{m-1} A(T_i)}$ be the metric projection of $H$ onto $\bigcap_{i=0}^{m-1} A(T_i)$. Let $\{x_n\}$ be a sequence in $C$ generated by*

$$\begin{aligned} x_1 &\in C \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n) y_n \\ y_n &= \beta_n x_n + (1-\beta_n) \frac{1}{m} \sum_{i=0}^{m-1} T_i x_n, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that $liminf_{n\to\infty} \beta_n (1-\alpha_n)(1-\beta_n) > 0$. Then, the sequence $\{x_n\}$ converges weakly to $\bar{x} = P_{\bigcap_{i=0}^{m-1} A(T_i)} w$. Additionally, if $C$ is closed and convex then $\{x_n\}$ converges weakly to

$$\bar{x} = P_{\bigcap_{i=0}^{m-1} Fix(T_i)} w.$$

**Proof.** A generalized hybrid mapping is nonexpansive mapping by taking $\alpha = 0$ and $\beta = 1$. Thus, by using Theorem 3.1 and 3.2, we got the result.

**Theorem 4.8:** *Let $H$ be a real Hilbert space and let $C$ be a nonempty convex subset of $H$. Let $T_i: C \to C$ be a nonspreading mapping for each $i = 0,1,2,\ldots,m-1$ with $A(T_i) \neq \emptyset$ and let $P_{\bigcap_{i=0}^{m-1} A(T_i)}$ be the metric projection of $H$ onto $\bigcap_{i=0}^{m-1} A(T_i)$. Let $\{x_n\}$ be a sequence in $C$ generated by*

$$\begin{array}{lll} x_1 & \in & C \\ x_{n+1} & = & \alpha_n x_n + (1-\alpha_n) y_n \\ y_n & = & \beta_n x_n + (1-\beta_n) \dfrac{1}{m} \displaystyle\sum_{i=0}^{m-1} T_i x_n, \end{array}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that $liminf_{n\to\infty}\beta_n(1-\alpha_n)(1-\beta_n) > 0$. Then, the sequence $\{x_n\}$ converges weakly to $\bar{x} = P_{\bigcap_{i=0}^{m-1} A(T_i)} w$. Additionally, if $C$ is closed and convex then $\{x_n\}$ converges weakly to $\bar{x} = P_{\bigcap_{i=0}^{m-1} Fix(T_i)} w$.

**Proof.** A generalized hybrid mapping is nonspreading mapping by taking $\alpha = 2$ and $\beta = 1$. Thus, by using Theorem 4.1 and 4.2, we got the result.

**Theorem 4.9:** *Let $H$ be a real Hilbert space and let $C$ be a nonempty convex subset of $H$. Let $T^i : C \to C$ be a hybrid mapping for each $i = 0,1,2,\dots,m-1$ with $A(T_i) \neq \emptyset$ and let $P_{\bigcap_{i=0}^{m-1} A(T_i)}$ be the metric projection of $H$ onto $\bigcap_{i=0}^{m-1} A(T_i)$. Let $\{x_n\}$ be a sequence in $C$ generated by*

$$\begin{array}{lll} x_1 & \in & C \\ x_{n+1} & = & \alpha_n x_n + (1-\alpha_n) y_n \\ y_n & = & \beta_n x_n + (1-\beta_n) \dfrac{1}{m} \displaystyle\sum_{i=0}^{m-1} T_i x_n, \end{array}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that $\lim\ inf_{n\to\infty}\beta_n(1-\alpha_n)(1-\beta_n) > 0$. Then, the sequence $\{x_n\}$ converges weakly to $\bar{x} = P_{\bigcap_{i=0}^{m-1} A(T_i)} w$. Additionally, if $C$ is closed and convex then $\{x_n\}$ converges weakly to

$$\bar{x} = P_{\bigcap_{i=0}^{m-1} Fix(T_i)} w.$$

**Proof.** A generalized hybrid mapping is hybrid mapping by taking $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. Thus, by using Theorem 3.1 and 3.2, we got the result.

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# A new extension of Beta function and its properties

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**Abstract:** In this paper, we aim to introduce a new extension of beta function and study its important properties. Using this definition, new extended hypergeometric and confluent hypergeometric functions are obtained. Further, some hybrid representations of this extended beta function are derived which include some well-known special functions and polynomials.

**Keywords:** Gamma function, Beta function, Hypergeometric function, Confluent hypergeometric function, Beta Distribution

## 1. Introduction and preliminaries

Extending well known special functions have been an active and interesting area of research. For the extension of beta and other special functions, several papers are published in literature (see [1]-[13], [15], [17]) due to their never-ending applications. Following up with the investigation, we define here a new extension of beta function and derive its integral representations, summation formula and some other relations. Further, we obtain beta distribution and some statistical formulas. Finally, using our definition of extended beta function $B_{p,q}^{\lambda}(\eta_1, \eta_2)$, we extend the definitions of hypergeometric and confluent hypergeometric functions. At last, we obtain connections of extended beta function with other special functions and polynomials from application viewpoint.

Throughout the paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers, positive real numbers and non positive integers respectively, and let $\mathbb{R}_0^+ := \mathbb{R} \cup \{0\}$.

**Definition 1.1** As is well known, the Gamma function $\Gamma(z)$ developed by Euler [1] with the intent to extend the factorials to values between the integers is defined by the definite integral

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\, dt \quad (\Re(z) > 0), \tag{1.1}$$

Among various extensions of gamma function, we mention here the extended gamma function [6] defined by Chaudhry and Zubair

$$\Gamma_p(z) =: \int_0^\infty t^{z-1} \exp\left(-t - \frac{p}{t}\right) dt \quad (\Re(p) > 0). \tag{1.2}$$

**Definition 1.2** Euler introduced the beta function (see [1]) for a pair of complex numbers $\eta_1$ and $\eta_2$ with positive real part through the integral

$$B(\eta_1, \eta_2) = \int_0^1 t^{\eta_1 - 1} (1-t)^{\eta_2 - 1} dt \quad = \frac{\Gamma(\eta_1)\Gamma(\eta_2)}{\Gamma(\eta_1 + \eta_2)} = \frac{(\eta_1 - 1)!\ (\eta_2 - 1)!}{(\eta_1 + \eta_2 - 1)!}. \tag{1.3}$$

In 1997, Chaudhry et al. [3] defined an extension of beta function as

$$B_p(\eta_1, \eta_2) = \int_0^1 t^{\eta_1 - 1} (1-t)^{\eta_2 - 1} \exp\left(-\frac{p}{t(1-t)}\right) dt, \quad (\Re(p) > 0). \tag{1.4}$$

Shadab et al. [15] introduced an interesting extension of the beta function involving the Mittag Leffler function defined as

$$B_p^\lambda(\eta_1, \eta_2) = \int_0^1 t^{\eta_1 - 1} (1-t)^{\eta_2 - 1} E_\lambda\left(-\frac{p}{t(1-t)}\right) dt \quad (\Re(p) > 0\ ;\ \lambda \in \mathbb{R}_0^+), \tag{1.5}$$

where $E_\lambda(.)$ is the classical Mittag Leffler function defined as

$$E_\lambda(x) = \sum_{n=0}^\infty \frac{x^n}{\Gamma(\lambda n + 1)}. \tag{1.6}$$

Note that, by putting $\lambda = 1$, the above definition corresponds to the extended beta function [3] and on putting $\lambda = 1$ and $p = 0$, we get the basic beta function given by (1.3).

They also studied the extended form of beta distribution [15]

$$f(t) = \begin{cases} \dfrac{1}{B_p^\lambda(\eta, \beta)} t^{\eta - 1} (1-t)^{\beta - 1} E_\lambda\left(-\dfrac{p}{t(1-t)}\right) & 0 < t < 1 \\ 0 & otherwise\,. \end{cases} \tag{1.7}$$

**Definition 1.3** The classical Gauss's hypergeometric function is defined by

$${}_2F_1\left[\begin{matrix} a, & b; \\ c; & \end{matrix} \quad z\right] = \sum_{n=0}^\infty \frac{(a)_n\ (b)_n}{(c)_n} \frac{z^n}{n!} = \ {}_2F_1(a, b; c;\ z), \tag{1.8}$$

where $(a)_n$ $(a \in \mathbb{C})$ is the well known Pochhammer symbol.

It is a particular case of the generalized hypergeometric series ${}_pF_q\ (p, q \in \mathbb{N}_0)$ defined by

$$ {}_pF_q\left[\begin{matrix} a_1, \dots, a_p; \\ b_1, \dots, b_q; \end{matrix} \quad z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}. $$

The confluent hypergeometric function (see [1]) is given by the series representation

$$ {}_1F_1(a; b;\ z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}. \tag{1.9} $$

The extended hypergeometric and confluent hypergeometric functions [4] are defined respectively by

$$ F_p(\eta_1, \eta_2, \eta_3; z) = \sum_{n=0}^{\infty} (\eta_1)_n \frac{B_p(\eta_2 + n, \eta_3 - \eta_2)}{B(\eta_2, \eta_3 - \eta_2)} \frac{z^n}{n!} $$
$$ (p \geq 0, \Re(\eta_3) > \Re(\eta_2) > 0 \text{ and } |z| < 1) \tag{1.10} $$

and

$$ \Phi_p(\eta_2; \eta_3; z) = \sum_{n=0}^{\infty} \frac{B_p(\eta_2 + n, \eta_3 - \eta_2)}{B(\eta_2, \eta_3 - \eta_2)} \frac{z^n}{n!}, $$
$$ (p \geq 0, \Re(\eta_3) > \Re(\eta_2) > 0 \text{ and } |z| < 1). \tag{1.11} $$

Their integral representations are:

$$ F_p(\eta_1, \eta_2, \eta_3; z) = \frac{1}{B(\eta_2, \eta_2 - \eta_3)} \int_0^1 t^{\eta_2 - 1}(1-t)^{\eta_3 - \eta_2 - 1} (1 - zt)^{-\eta_1} \exp\left(-\frac{p}{t(1-t)}\right) dt $$
$$ (p > 0;\ p = 0 \text{ and } |z| < 1;\ \Re(\eta_3) > \Re(\eta_2) > 0), \tag{1.12} $$

and

$$ \phi_p(\eta_2; \eta_3; z) = \frac{1}{B(\eta_2, \eta_2 - \eta_3)} \int_0^1 t^{\eta_2 - 1}(1-t)^{\eta_3 - \eta_2 - 1} \exp\left(zt - \frac{p}{t(1-t)}\right) dt, $$
$$ (p > 0;\ p = 0 \text{ and } \Re(\eta_3) > \Re(\eta_2) > 0). \tag{1.13} $$

## 2. A new extension of Beta function

Here, we introduce a new extension of the generalized Beta function $B_p^{\lambda}(\eta_1, \eta_2)$ in (1.5) and obtain its various properties and representations.

**Definition 2.1** We define a new extension of beta function as

$$ B_{p,q}^{\lambda}(\eta_1, \eta_2) = \int_0^1 t^{\eta_1 - 1} (1-t)^{\eta_2 - 1} E_{\lambda}\left(-\frac{p}{t} - \frac{q}{(1-t)}\right) dt \tag{2.1} $$

$$(\Re(p) \geq 0, \Re(q) \geq 0;\ \lambda > 0),$$

where $E_\lambda$ is the Mittag-Leffler function.

Remark 2.1 Note that for $\lambda = 1$, (2.1) reduce to definition [7]. For $p = q$ and $p = 0 = q$, (2.1) reduce to the extended beta function [15] and the classical beta function given by (1.3) respectively.

### 2.1 Integral Representations of $B_{p,q}^{\lambda}(\eta_1, \eta_2)$

**Theorem 1** The following integral representations holds:

$$B_{p,q}^{\lambda}(\eta_1, \eta_2) = 2\int_0^{\frac{\pi}{2}} \cos^{2\eta_1 - 1}\theta \sin^{2\eta_2 - 1}\theta\, E_\lambda\left[-\frac{p}{\cos^2\theta} - \frac{q}{\sin^2\theta}\right] d\theta, \tag{2.2}$$

$$B_{p,q}^{\lambda}(\eta_1, \eta_2) = \int_0^{\infty} \frac{u^{\eta_1 - 1}}{(1+u)^{\eta_1+\eta_2}} E_\lambda\left(-\frac{p(1+u)}{u} - \frac{q}{1+u}\right) du, \tag{2.3}$$

$$B_{p,q}^{\lambda}(\eta_1, \eta_2) = 2^{1-\eta_1-\eta_2} \times \int_{-1}^{1} (1+u)^{\eta_1 - 1} (1 - u)^{\eta_2} \frac{u^{\eta_1 - 1}}{(1+u)^{\eta_1+\eta_2-1}} E_\lambda\left(-\frac{2(p+q) + 2(q-p)u}{1-u^2}\right) du, \tag{2.4}$$

$$B_{p,q}^{\lambda}(\eta_1, \eta_2) = (c-a)^{1-\eta_1-\eta_2} \times \int_a^c (u-a)^{\eta_1 - 1}(c - u)^{\eta_2 - 1} E_\lambda\left[-\frac{c-a}{(u-a)(c-u)}((q-p)u + (pc - qa))\right] du,$$

$$(\Re(p) > 0, \Re(q) > 0;\ p \geq 0, q \geq 0;\ \Re(\eta_1) > 0, \Re(\eta_2) > 0). \tag{2.5}$$

**Proof.** Let $t = cos^2\,\theta,\ t = \frac{u}{1+u},\ t = \frac{1+u}{2},\ t = \frac{u-a}{c-a}$ respectively in equations (2.1), we obtain the above representations.

Remark 2.2 The above results retrieve the corresponding representations in [15] and [7] by taking $p = q$ and $\lambda = 1$ respectively. Further for $p = 0 = q$ and $\lambda = 1$, the results reduce to some well-known results for the beta function $B(\eta_1, \eta_2)$.

## 3. Properties of $B_{p,q}^{\lambda}(\eta_1, \eta_2)$

In this section we obtain some interesting relations, summation formulas and product formulas for the generalized beta function $B_{p,q}^{\lambda}(\eta_1, \eta_2)$.

**Theorem 2** The extended beta function satisfies the following functional relation:

$$B_{p,q}^{\lambda}(\eta_1 + 1, \eta_2) + B_{p,q}^{\lambda}(\eta_1, \eta_2 + 1) = B_{p,q}^{\lambda}(\eta_1, \eta_2). \tag{3.1}$$

**Proof.** Using (2.1) in the l.h.s. of (3.1), we get

$$B_{p,q}^{\lambda}(\eta_1+1,\eta_2)+B_{p,q}^{\lambda}(\eta_1,\eta_2+1)$$
$$=\int_0^1 \{\, t^{\eta_1}\,(1-t)^{\eta_2-1}+t^{\eta_1-1}\,(1-t)^{\eta_2}\,\}\, E_\lambda\left(-\frac{p}{t}-\frac{q}{1-t}\right)\, dt,$$

and a little manipulation leads us to the desired result.

**Remark 3.1** Again, the case $p=q$ and $p=0=q$ of equation (3.1) reduces to the corresponding result in [15] and some known relations.

**Theorem 3** The following summation formula is valid for $B_{p,q}^{\lambda}(\eta_1,\eta_2)$:

$$B_{p,q}^{\lambda}(\eta_1,1-\eta_2)=\sum_{n=0}^{\infty}\frac{(\eta_2)_n}{n!}\,B_{p,q}^{\lambda}(\eta_1+n,1)\quad(\Re(p)>0,\Re(q)>0). \tag{3.2}$$

**Proof.** To prove above result, we make use of the generalized binomial theorem defined as

$$(1-t)^{-\eta_2}=\sum_{n=0}^{\infty}(\eta_2)_n\,\frac{t^n}{n!},\quad(|t|<1).$$

Therefore, from definition (2.1), we can write

$$B_{p,q}^{\lambda}(\eta_1,1-\eta_2)=\int_0^1\sum_{n=0}^{\infty}(\eta_2)_n\,\frac{t^{\eta_1+n-1}}{n!}\,E_\lambda\left(-\frac{p}{t}-\frac{q}{1-t}\right)\,dt.$$

Now by interchanging the order of integration and summation, we can easily obtain the desired formula.

**Theorem 4** For $\Re(p)>0$, $\Re(q)>0$, the following infinite summation formula holds:

$$B_{p,q}^{\lambda}(\eta_1,\eta_2)=\sum_{n=0}^{\infty}B_{p,q}^{\lambda}(\eta_1+n,\eta_2+1). \tag{3.3}$$

**Proof.** Using the relation

$$(1-t)^{\eta_2-1}=(1-t)^{\eta_2}\sum_{n=0}^{\infty}t^n,$$

We obtain

$$B_{p,q}^{\lambda}(\eta_1,\eta_2)=\int_0^1(1-t)^{\eta_2}\sum_{n=0}^{\infty}t^{\eta_1+n-1}E_\lambda\left(-\frac{p}{t}-\frac{q}{1-t}\right)\,dt.$$

Interchanging the order of integration and summation in the last expression leads us to the desired result.

**Theorem 5** The following relation holds true:

$$B_{p,q}^{\lambda}(\eta,-\eta-n)=\sum_{k=0}^{n}\binom{n}{k}B_{p,q}^{\lambda}(\eta+k,-\eta-k)\quad(n\in\mathbb{N}_0). \tag{3.4}$$

**Proof.** We have

$$B_{p,q}^{\lambda}(\eta_1+1,\eta_2)+B_{p,q}^{\lambda}(\eta_1,\eta_2+1)=B_{p,q}^{\lambda}(\eta_1,\eta_2).$$

On substituting $\eta_1=\eta$ and $\eta_2=-\eta-n$ above, we arrive at

$$B_{p,q}^{\lambda}(\eta,-\eta-n)=B_{p,q}^{\lambda}(\eta,-\eta-n+1)+B_{p,q}^{\lambda}(\eta+1,-\eta-n).$$

Writing this formula recursively with $n=1,2,3.\ldots,$, we obtain

$$B_{p,q}^{\lambda}(\eta,-\eta-1)=B_{p,q}^{\lambda}(\eta,-\eta)+B_{p,q}^{\lambda}(\eta+1,-\eta-1),$$

$$B_{p,q}^{\lambda}(\eta,-\eta-2)=B_{p,q}^{\lambda}(\eta,-\eta)+2B_{p,q}^{\lambda}(\eta+1,-\eta-1)+B_{p,q}^{\lambda}(\eta+2,-\eta-2),$$

and so on. By continuing the process, we arrive at (3.4).

In statistical distribution theory, gamma and beta functions have been used extensively. We now define the beta distribution of (2.1), and obtain its mean, variance and moment generating function.

For $B_{p,q}^{\lambda}(\eta_1,\eta_2)$, the beta distribution is given by

$$f(t)=\begin{cases}\dfrac{1}{B_{p,q}^{\lambda}(\eta_1,\eta_2)}t^{\eta_1-1}(1-t)^{\eta_2-1}E_{\lambda}\left(-\dfrac{p}{t}-\dfrac{q}{(1-t)}\right) & (0<t<1),\\ 0 & otherwise\,.\end{cases} \tag{3.5}$$

For any real number $\nu$, we have the $d$th moment of a random variable $X$ as

$$\mathbb{E}(X^{\nu})=\frac{B_{p,q}^{\lambda}(\eta_1+\nu,\eta_2)}{B_{p,q}^{\lambda}(\eta_1,\eta_2)}$$

$$(\eta_1,\eta_2\in\mathbb{R};\ p,q,\lambda\in\mathbb{R}^{+}). \tag{3.6}$$

When $\nu=1$, we get the mean as a particular case of (3.6)

$$\mu=\mathbb{E}(X)=\frac{B_{p,q}^{\lambda}(\eta_1+1,\eta_2)}{B_{p,q}^{\lambda}(\eta_1,\eta_2)}, \tag{3.7}$$

and the variance of the distribution is defined by

$$\sigma^2 = \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2 = \frac{B_{p,q}^{\lambda}(\eta_1,\eta_2)\, B_{p,q}^{\lambda}(\eta_1+2,\eta_2) - \left\{B_{p,q}^{\lambda}(\eta_1+1,\eta_2)\right\}^2}{\left\{B_{p,q}^{\lambda}(\eta_1,\eta_2)\right\}^2}. \tag{3.8}$$

The moment generating function of the distribution is defined by

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, \mathbb{E}(X^n) = \frac{1}{B_{p,q}^{\lambda}(\eta_1,\eta_2)} \sum_{n=0}^{\infty} B_{p,q}^{\lambda}(\eta_1+n,\eta_2) \frac{t^n}{n!}. \tag{3.9}$$

The cumulative distribution is given by

$$F(x) = \frac{B_{z,p,q}^{\lambda}(\eta_1,\eta_2)}{B_{p,q}^{\lambda}(\eta_1,\eta_2)} \tag{3.10}$$

where

$$B_{x,p,q}^{\lambda}(\eta_1+1,\eta_2) = \int_0^x t^{\eta_1-1}(1-t)^{\eta_2-1}\, E_{\lambda}\left(-\frac{p}{t} - \frac{q}{(1-t)}\right) dt$$
$$(p > 0, q > 0, \lambda > 0, -\infty < \eta_1, \eta_2 < \infty) \tag{3.11}$$

is the extended incomplete beta function.

## 4. Generalization of Extended Hypergeometric and Confluent Hypergeometric functions

Here, we introduce a generalization of extended hypergeometric and confluent hypergeometric functions in terms of $B_{p,q}^{\lambda}(\eta_1,\eta_2)$.

$$F_{p,q}^{\lambda}(\eta_1,\eta_2;\eta_3;z) = \sum_{n=0}^{\infty} (\eta_1)_n \frac{B_{p,q}^{\lambda}(\eta_2+n,\eta_3-\eta_2)}{B(\eta_2;\eta_3-\eta_2)} \frac{z^n}{n!},$$
$$(p \geq 0, q \geq 0, |z| < 1, \lambda > 0, \Re(\eta_3) > \Re(\eta_2) > 0) \tag{4.1}$$

and

$$\Phi_{p,q}^{\lambda}(\eta_2;\eta_3;z) = \sum_{n=0}^{\infty} \frac{B_{p,q}^{\lambda}(\eta_2+n,\eta_3-\eta_2)}{B(\eta_2;\eta_3-\eta_2)} \frac{z^n}{n!}.$$
$$(p > 0, q > 0, \lambda > 0, \Re(\eta_3) > \Re(\eta_2) > 0) \tag{4.2}$$

$F_{p,q}^{\lambda}(\eta_1,\eta_2;\eta_3;z)$ and $\Phi_{p,q}^{\lambda}(\eta_2;\eta_3;z)$ are the further generalizations of the extended Gauss hypergeometric function and extended confluent hypergeometric function given by (1.10) and (1.11) respectively.

**4.1 Integral Representations**

**Theorem 6** The following integral representations for the extended hypergeometric $F_{p,q}^{\lambda}(\eta_1,\eta_2;\eta_3;z)$ and confluent hypergeometric function $\Phi_{p,q}^{\lambda}(\eta_1,\eta_2;\eta_3;z)$ holds true:

$$F_{p,q}^{\lambda}(\eta_1,\eta_2;\eta_3;z) = \frac{1}{B(\eta_2,\eta_3-\eta_2)} \times \int_0^1 t^{\eta_2-1}(1-t)^{\eta_3-\eta_2-1} E_\lambda\left(-\frac{p}{t}-\frac{q}{1-t}\right) \sum_{n=0}^{\infty} (\eta_1)_n \frac{(zt)^n}{n!}\, dt,$$

$$(p>0, q>0;\ \lambda>0;\ p=0, q=0\ and\ |z|<1;\ \Re(\eta_3)>\Re(\eta_2)>0). \tag{4.3}$$

$$F_{p,q}^{\lambda}(\eta_1,\eta_2;\eta_3;z) = \frac{1}{B(\eta_2,\eta_3-\eta_2)} \times \int_0^1 t^{\eta_2-1}(1-t)^{\eta_3-\eta_2-1}(1-zt)^{-\eta_1} E_\lambda\left(-\frac{p}{t}-\frac{q}{1-t}\right) dt,$$

$$(p>0, q>0;\ \lambda>0;\ p=0, q=0\ and\ |\arg(1-z)|<\pi;\ \Re(\eta_3)>\Re(\eta_2)>0). \tag{4.4}$$

$$F_{p,q}^{\lambda}(\eta_1,\eta_2;\eta_3;z) = \frac{1}{B(\eta_2,\eta_3-\eta_2)} \times \int_0^\infty u^{\eta_2-1}(1+u)^{\eta_1-\eta_3}[u(1-z)]^{-\eta_1} E_\lambda\left(-\frac{p(1+u)}{u}-q(1+u)\right) du,$$

$$(p>0, q>0;\ \lambda>0;\ p=0, q=0\ and\ |\arg(1-z)|<\pi;\ \Re(\eta_3)>\Re(\eta_2)>0). \tag{4.5}$$

$$F_{p,q}^{\lambda}(\eta_1,\eta_2;\eta_3;z) = \frac{2}{B(\eta_2,\eta_3-\eta_2)} \times \int_0^{\frac{\pi}{2}} \frac{\sin^{2\eta_2-1}v \cos^{2\eta_3-2\eta_2-1} v}{(1-z\sin^2 v)^{\eta_1}}\ E_\lambda(-p\sec^2 v - q\csc^2 v)\, dv,$$

$$(p>0, q>0;\ \lambda>0;\ p=0, q=0\ and\ |\arg(1-z)|<\pi;\ \Re(\eta_3)>\Re(\eta_2)>0). \tag{4.6}$$

$$\Phi_{p,q}^{\lambda}(\eta_2;\eta_3;z) = \frac{\exp(zt)}{B(\eta_2,\eta_3-\eta_2)} \times \int_0^1 t^{\eta_2-1}(1-t)^{\eta_3-\eta_2-1} E_\lambda\left(-\frac{p}{t}-\frac{q}{1-t}\right) dt,$$

$$(p>0, q>0;\ \lambda>0;\ \Re(\eta_3)>\Re(\eta_2)>0). \tag{4.7}$$

$$\Phi_{p,q}^{\lambda}(\eta_2;\eta_3;z) = \frac{\exp(z)}{B(\eta_2,\eta_3-\eta_2)} \times \int_0^1 t^{\eta_2-1}(1-t)^{\eta_3-\eta_2-1} \exp(-zt)\, E_\lambda\left(-\frac{p}{t}-\frac{q}{1-t}\right) dt.$$

$$(p>0, q>0;\ \lambda>0;\ \Re(\eta_3)>\Re(\eta_2)>0). \tag{4.8}$$

**Proof.** We can obtain (4.3) by using the definition (2.1) in (4.1). The integral (4.4) can be obtained by using the binomial expansion

$$(1-zt)^{-\eta_1} = \sum_{n=0}^{\infty} (\eta_1)_n \frac{(zt)^n}{n!}$$

in (4.3). By substituting $t = \frac{u}{1+u}$, $t = \sin^2 v$ in (4.4), we obtain (4.5) and (4.6) respectively. By using a similar approach, we can easily establish the representations (4.7) and (4.8).

**Remark 4.1** The case $p = q$ and $\lambda = 1$ in equations (4.3) -(4.8) leads to the corresponding results in [3]. For $p = 0 = q$ and $\lambda = 1$, we get basic hypergeometric and confluent hypergeometric function [1].

## 5 Differentiation formulas for $F_{p,q}^{\lambda}(\eta_1, \eta_2; \eta_3;\ z)$ and $\Phi_{p,q}^{\lambda}(\eta_1, \eta_2; \eta_3;\ z)$

By differentiating (4.1) and (4.2), we obtain some differentiation formulas with the help of the formula:

$$B(\eta_2, \eta_3 - \eta_1) = \frac{\eta_3}{\eta_2}\ B(\eta_2 + 1, \eta_3 - \eta_2). \tag{5.1}$$

**Theorem 7** The following differentiation formulas are true:

$$\frac{d}{dz}\left\{F_{p,q}^{\lambda}(\eta_1, \eta_2; \eta_3; z)\right\} = \frac{\eta_1 \eta_2}{\eta_3}\ F_{p,q}^{\lambda}(\eta_1 + 1, \eta_2 + 1; \eta_3 + 1;\ z). \tag{5.2}$$

$$\frac{d^r}{dz^r}\left\{F_{p,q}^{\lambda}(\eta_1, \eta_2; \eta_3; z)\right\} = \frac{(\eta_1)_r (\eta_2)_r}{(\eta_3)_r}\ F_{p,q}^{\lambda}(\eta_1 + r, \eta_2 + r; \eta_3 + r;\ z) \quad (r \in \mathbb{N}_0). \tag{5.3}$$

$$\frac{d^r}{dz^r}\left\{\Phi_{p,q}^{\lambda}(\eta_2; \eta_3; z)\right\} = \frac{(\eta_2)_r}{(\eta_3)_r}\ \Phi_{p,q}^{\lambda}(\eta_2 + r; \eta_3 + r;\ z) \quad (r \in \mathbb{N}_0). \tag{5.4}$$

**Proof.** By differentiating (4.1) with respect to $z$, we get

$$\frac{d}{dz}\left\{F_{p,q}^{\lambda}(\eta_1, \eta_2; \eta_3; z)\right\} = \sum_{n=1}^{\infty} \frac{B_{p,q}^{\lambda}(\eta_2 + n, \eta_3 - \eta_2)}{B(\eta_2, \eta_3 - \eta_2)}\ (\eta_1)_n\ \frac{z^{n-1}}{(n-1)!}.$$

On replacing $n$ by $n+1$ and using (4.4), we easily get (5.2). A recursive process of this establishes (5.3). In a similar way, we can obtain (5.4).

**Remark 5.1** The case $p = q$ and $\lambda = 1$ in equations (4.3)-(4.8) leads to the corresponding results in [3]. For $p = 0 = q$ and $\lambda = 1$, we get corresponding formulas for hypergeometric and confluent hypergeometric function (see [1]).

### 5.1 Transformation formulas

The following formulas for the extended hypergeometric and confluent hypergeometric function holds true:

$$F_{p,q}^{\lambda}(\eta_1,\eta_2;\eta_3;\ z) = (1-z)^{-\alpha}\ F_{p,q}^{\lambda}\left(\eta_1,\eta_3-\eta_2;\eta_3;\ -\frac{z}{1-z}\right). \tag{5.5}$$

$$(\lambda \in \mathbb{R}^+;\ p,q \in \mathbb{R}_0^+;\ |z|<1;\ \Re(\eta_3) > \Re(\eta_2) > 0).$$

$$F_{p,q}^{\lambda}\left(\eta_1,\eta_2;\eta_3;\ 1-\frac{1}{z}\right) = z^{\alpha}\ F_{p,q}^{\lambda}(\eta_1,\eta_3-\eta_2;\eta_3;\ 1-z). \tag{5.6}$$

$$(\lambda \in \mathbb{R}^+;\ p,q \in \mathbb{R}_0^+;\ |z|<1;\ \Re(\eta_3) > \Re(\eta_2) > 0).$$

$$F_{p,q}^{\lambda}\left(\eta_1,\eta_2;\eta_3;\ \frac{z}{1+z}\right) = (1+z)^{\alpha}\ F_{p,q}^{\lambda}(\eta_1,\eta_3-\eta_2;\eta_3;\ -z). \tag{5.7}$$

$$(\lambda \in \mathbb{R}^+;\ p,q \in \mathbb{R}_0^+;\ |z|<1;\ \Re(\eta_3) > \Re(\eta_2) > 0).$$

$$\Phi_{p,q}^{\lambda}(\eta_2,\eta_3;\ z) = e^{z}\ \Phi_{p,q}^{\lambda}(\eta_3-\eta_2;\eta_3;\ -z). \tag{5.8}$$

**Proof.** Replacing $t$ by $1-t$ in (4.4) and with the help of expression

$$[1-z(1-t)]^{-\eta_1} = (1-z)^{-\eta_1}\left(1+\frac{z}{1-z}t\right)^{-\eta_1},$$

we have

$$F_{p,q}^{\lambda}(\eta_1,\eta_2;\eta_3;z) = \frac{(1-z)^{-\eta_1}}{B(\eta_2,\eta_3-\eta_2)} \times \int_0^1 t^{\eta_2-1}(1-t)^{\eta_3-\eta_2-1}\left(1+\frac{z}{1-z}t\right)^{-\eta_1} E_{\lambda}\left(-\frac{p}{t}-\frac{q}{1-t}\right)dt, \tag{5.9}$$

$$(p>0, q>0;\ \lambda>0;\ p=0, q=0\ and\ |\arg(1-z)|<\pi;\ \Re(\eta_3) > \Re(\eta_2) > 0)$$

which easily proves (5.5). Replacing $z$ by $\left(1-\frac{1}{z}\right)$ and $\left(\frac{z}{1+z}\right)$ in (5.5) yields (5.6) and (5.7) respectively. Now the formula (5.8) can be obtained by following (4.7) and (4.8).

### 6 Representations for $B_{p,q}^{\lambda}(\eta_1,\eta_2)$

In this section we obtain certain connections of the beta function (2.1) in terms of other special functions and polynomials. The results obtained here are interesting and can further be applied to other extensions of beta functions.

• (Generalized hypergeometric representation).

Since we have the relation [16]

$$E_{\lambda,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\lambda n+\beta)} \frac{z^n}{n!} = \frac{1}{\Gamma(\beta)} \, {}_qF_\lambda \left[\Delta(q;\gamma);\ \Delta(\lambda;\beta);\quad \frac{q^q z}{\lambda^\lambda}\right], \tag{6.1}$$

where, $\Delta(q,\gamma)$ is a q-tuple $\frac{\gamma}{q}, \frac{\gamma+1}{q}, \ldots, \frac{\gamma+q-1}{q}$.

In particular, we have

$$E_{\lambda,1}^{1,1}(z) = E_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n+1)} = {}_1F_\lambda \left[\Delta(1;1);\ \Delta(\lambda,1);\quad \frac{z}{\lambda^\lambda}\right]. \tag{6.2}$$

Now using (6.2) in (2.1), we have

$$B_{p,q}^{\lambda}(\eta_1,\eta_2) = \int_0^1 t^{\eta_1-1} (1-t)^{\eta_2-1} \times {}_1F_\lambda \left[\Delta(1;1);\ \Delta(\lambda,1);\quad \frac{1}{\lambda^\lambda}\left(-\frac{p}{t}-\frac{q}{(1-t)}\right)\right] dt, \tag{6.3}$$

which can be written as

$$B_{p,q}^{\lambda}(\eta_1,\eta_2) = \int_0^1 \frac{u^{\eta_1-1}}{(1+u)^{\eta_1+\eta_2}} \times {}_1F_\lambda \left[\Delta(1;1);\ \Delta(\lambda,1);\quad \frac{(1+u)[-p-qu]}{u\,\lambda^\lambda}\right] du. \tag{6.4}$$

• (Fox $H$- function representation)

The following relation between $B_{p,q}^{\lambda}(\eta_1,\eta_2)$ and Fox-H function holds true:

$$B_{p,q}^{\lambda}(\eta_1,\eta_2) = \int_0^1 t^{\eta_1-1} (1-t)^{\eta_2-1} H_{0,2}^{1,0}\left[\frac{p}{t}+\frac{q}{1-t} \;\middle|\; (0;1);\ (0,1),(0,\lambda)\right] dt. \tag{6.5}$$

• (Bessel function and Laguerre polynomial representation)

We obtain a relationship between the generalized beta function $B_{p,q}^{\lambda}(\eta_1,\eta_2)$, Laguerre polynomials $L_n^{\beta}(\lambda;x)$ and Bessel function $J_\beta(x)$.
Since we have the relation [14]

$$\Gamma(1+\beta)\,(xt)^{\frac{\beta}{2}} E_\lambda(t)\, J_\beta(2\sqrt{xt}) = \sum_{n=0}^{\infty} \frac{L_n^{\beta}(\lambda;x)\, t^n}{(1+\beta)_n}, \tag{6.6}$$

we can write

$$B_{p,q}^{\lambda}(\eta_1,\eta_2)=\frac{1}{x^{\beta/2}}\int_0^1 \frac{t^{\eta_1-1}\,(1-t)^{\eta_2-1}}{J_\beta\,(2\sqrt{x\mu})}\left(\sum_{n=0}^{\infty}\frac{L_n^{\beta}(\lambda;x)\,(\mu)^{n-\beta/2}}{(\beta+n)!}\right)dt, \tag{6.7}$$

where, $\mu=-\frac{p}{t}-\frac{q}{1-t}$.

## 7 Discussion and Conclusion

In this paper, we have introduced a new extension of beta function which seems to be interesting since by being specific on parameters, some well-known definitions of the beta function can be retrieved. Further, it is shown that this extended beta function can be represented in terms of other polynomials and special functions. As the beta and hypergeometric functions have so many applications in the literature, a lot more relationships with different functions can be found which may be potentially useful for further research.

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# Time Dependent Perturbation Theory in Two level Quantum System and Its Applications

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**Abstract:** The calculation of the transition probabilities between two stationary energy states under a time dependent potential is one of the problems that can be investigated via time dependent perturbation theory. It explains how states of an unperturbed system vary with time under small perturbations. In this work we use analytical and semi analytical methods to study two levels of quantum mechanical systems using time dependent perturbation theory. In the first part of the work we focus on exact solutions and reconstruction techniques, we built the potential function from a given quantum amplitude, we obtained the potential for some physical models proposed in the literature as realistic models compatible with experiments, for example Rabbi effect and the dynamics of the electron in an external magnetic field. The second part is devoted to finding the distance between two nearby quantum states in Hilbert space. We conclude this work by finding this distance for Landau-Zener effect as a two-level system.

**Keywords:** Quantum Mechanics; perturbation theory; Rabbi effect

## 1. Introduction

There are many problems in quantum physics that physicists couldn't solve exactly, especially the complex quantum systems. The physicists tried to solve those systems but instead of that they get a set of infinite differential equations [1, 2]. The Perturbation theory is a very powerful approximation in such systems, the quantum formulation which was invented by Paul Dirac to overcome such physical challenges [2, 3]. In this technique, we split the Hamiltonian of complex systems to an unperturbed system with known eigenfunctions of the Hamiltonian and a small perturbed Hamiltonian which represents the small disturbance of that system [3, 4]. The physical quantities related to the small disturbances, such as the energy levels and the eigenstate can be considered as corrections to the unperturbed system [4]. The Hamiltonian of the unperturbed system is considered as time independent, while the perturbation term can be either time dependent or not [5, 6]. The perturbation theory is classified in to two forms, depending in the potential disturbance and whether it depends to time or not. When the perturbation term is constant over time that leads to it being categorized as time independent perturbation theory. It modifies the behavior of the unperturbed eigenstates of the system and makes a difference between the stationary states for the perturbed system and the unperturbed system [7, 8]. The second class is the time dependent perturbation theory, here the perturbed part is no longer stationary and changes explicitly with time [1].

The most important example of phenomena that can be solved it exactly by quantum mechanics is the transition probabilities between two eigenstates, when this transition happens under the perturbation term called V($\vec{r}$,t), it varies with time or occurs suddenly [9, 10]. Time dependent perturbation theory is the best approximation way to deal with transition probabilities under V($\vec{r}$,t). It doesn't modify the unperturbed system states as the time independent perturbation but instead of that is varying from one state to another or it makes transitions under the perturbation [8, 11]. The Rabbi's effect is one of the applications related to time dependent perturbation theory. When the system absorbs plenty of photons it will be excited and after a period of time will emit those photons. This cyclic behavior between absorbing and emitting along the time is described as Rabi effect, it explains the cyclic oscillation of atoms between two quantum levels [1, 2, 6]. Another application is Fermi's Golden Rule. This rule it describes the rate of transition between two energy states under small perturbation [12, 13]. In this work, we will use two method to study two levels quantum mechanical systems in the time dependent perturbation theory. In section 3 we will use reconstruction techniques to build up the potential forms for some physical models which are proposed in the literature as an example of the Rabi effect. In section 4 we will find the distance between two nearby quantum states in Hilbert space for a two-level system. The last section is devoted to finding the distance between two nearby quantum states especially for Landau-Zener two level system. [1]

## 2. Master Equation for the Quantum Amplitude in Two Level Systems

In this section, we consider a two levels quantum system. The idea is to find the interaction potential from the given quantum amplitude forms. This will be a type of reconstruction differential equations. Let us start our study with a time evolution equation for amplitude, gives as $a_m(t)$ and satisfies the following coupled first order system of scheme.

$$i\hbar\frac{da_m}{dt} = \sum_k a_k(t)V_{mk}(t)e^{i\omega_{12}t}. \tag{1}$$

We represent the potential as a simple matrix with time dependent element:

$$V_{mk} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \\ & \end{bmatrix}$$

for $m, k = 1,2$.

This potential matrix has elements as functions of time. We supposed that we have two levels so the summation index is running from k=1 up to 2:

$$i\hbar\frac{da_1(t)}{dt} = a_1(t)V_{11}(t) + a_2(t)e^{i\omega_{12}t}V_{12}(t). \tag{2}$$

$$i\hbar\frac{da_2(t)}{dt} = a_1(t)e^{-i\omega_{21}t}V_{21}(t) + a_2(t)V_{22}(t). \tag{3}$$

We assume that $\omega_{12} = -\omega_{21} = \omega, V_{11}(t) = V_{22}(t) = V(t) \in \mathbb{R}$ as a real function and $V_{12}(t) = W(t), V_{21}(t) = \overline{W}(t)$ where "bar" indicates complex conjugate.

We reduce the pair of equations presented as above to the single second-order differential equations for amplitude, called Master equation:

$$a_1''(t) - a_1'(t)(-i\omega + 2V(t) + \frac{W'(t)}{W(t)}) - a_1(t)(i\omega V(t) + W^2(t) - V^2(t) + V'(t) - \frac{V(t)W'(t)}{W(t)}) = 0. \tag{4}$$

where W(t) is considered as a complex potential as a result, we suppose that $W(t) \equiv W = W_1 + iW_2$ and $V(t) \equiv V$.

The aim is to find $(W_1, W_2, V)$ when we have data about $a_1(t)$. Note that we can also find a similar equation for $a_2(t)$, but in this work we focus only on the first equation for $a_1(t)$.

## 3. Reconstruction Technique to Find the Potentials from Different Amplitude Models

In this section, we will reconstruct the potentials for different examples of amplitude. In labs we can find the data about the density of state (electron density) which is proportional to $|\psi|^2$. For two level systems, $|\psi|^2$ is proportional to amplitude $\sum_i |a_i(t)|^2$, it means when we measure the amplitude of transition (low density or high density) we are provided by the orbital forms as a question of how to describe the potential interaction when type of interaction or potentials are unknown. This is our main motivation to find V(t) and W(t) from master equation (4).

### 3.1 Harmonic Oscillator Amplitude

We start the reconstruction techniques with a simple harmonic oscillator quantum amplitude, the system can describe time oscillations of an electron in a uniform magnetic field $\vec{B} = B\hat{k}$ in "Z" representation of pauli's matrices. The general form of second order ordinary differential equation (ODE) of simple harmonic oscillator is given by:

$$a''(t) + a(t) = 0. \tag{5}$$

By comparing equation (5) with equation (4), we get the following equations:

$$2V + \frac{W_1' W_1 + W_2' W_2}{|W|^2} = 0, \tag{6}$$

$$\omega |W|^2 = W_1 W_2' - W_1' W_2, \tag{7}$$

$$V \left( \frac{W_1' W_1 + W_2' W_2}{|W|^2} \right) - |W|^2 + V^2 - V' = 1. \tag{8}$$

By dividing equation (7) by $W_1^2$, we obtain:

$$\omega \left( 1 + \frac{W_2^2}{W_1^2} \right) = \frac{d}{dt} \left( \frac{W_2}{W_1} \right),$$

by integrating the above equation, we obtain the relation between $W_1$ and $W_2$ as:

$$W_2 = W_1 \tan(\omega t). \tag{9}$$

Equation (9) helps us to rewrite the potential

$$W = \frac{W_1}{\cos(\omega t)} e^{i\omega t}$$

where $W_1$ can be a function of time. By plugging (6) and (9) in to the equation (8) we obtain:

$$V^2 + (W_1^2 + W_2^2) + V' = -1, \tag{10}$$

substitute (6) and (9) in (10), we get:

$$W_1'' = \frac{1}{4W_1(t)} (4\sec^2(\omega t) W_1^2 - 3\omega^2 \sec^2(\omega t) W_1^2 +$$

$$4\cos(2\omega t)\sec^2(\omega t) W_1^2 - \omega^2 \cos(2\omega t)\sec^2(\omega t) W_1^2 + 8\sec^2(\omega t) W_1^4 + 4\omega \tan(\omega t) W_1 W_1' + 6W_1'^2). \tag{11}$$

The equation (11) is a non-linear second order ODE. The general solution cannot be found easily. Instead of that, we can solve it by the iteration method. To do that, first we find the zeroth order approximated solution by omitting all terms $O(W_1^2)$ and then we replace that solution in the non-linear terms and we find $W_1^{(1)}$. After passing all these steps, we obtain:

$$W_1 = e^{\frac{1}{4}A(-2B+9A)} (C_1 + e^{BA} C_2) + t \int_0^t \frac{1}{8} e^{\frac{1}{4}B(-2A+9B)} H) dt, \tag{12}$$

where:

$$A = \frac{1}{2}\tan(\omega t) \left( \sqrt{9 + \frac{-8+\omega}{\omega^2}} - \sqrt{24 - \frac{4}{\omega^2} + \frac{(-4+\omega)\cos(2\omega t)}{\omega^2} + 9\sec^2(\omega t) + 12\tan(\omega t)} \right),$$

$$B = \sqrt{24 - \frac{4}{\omega^2} + \frac{(-4+\omega)\cos(2\omega t)}{\omega^2} + 9\sec^2(\omega t)\tan(\omega t)}.$$

$$H = (16e^{-AB+\frac{9B^2}{2}}(C_1 + e^{AB}C_2)^3\sec^2(\omega t) - 2(C_1 + e^{AB}C_2)(-4 + 3\omega^2 + (-4 + \omega^2)\cos(2\omega t))\sec^2(\omega t) + \frac{3(C_1(B(A' - 9B') + AB') - e^{AB}C_2(BA' + (A + 9B)B'))^2}{C_1 + e^{AB}C_2} + 4\omega\tan(\omega t)(-C_1(B(A' - 9B') + AB') + e^{AB}C_2(BA' + (A + 9B)B')).$$

We can use the same approximation technique to find V(t).

For that we use $W_1 = W_1^{(0)} + W_1^{(1)}$ and $W_1^2 = (W_1^{(0)} + W_1^{(1)})^2 = (W_1^{(0)})^2 + (W_1^{(1)})^2 + 2(W_1^{(0)})(W_1^{(2)}) << 1,$ we substitute (9) in (10):

$$V^2 + (W_1^2\sec^2(\omega t)) + V' = -1,$$

At zero approximation when the second term is omitted, we have:

$$V^{(0)2} + V^{(0)\prime} = -1,$$

The exact solution for this ODE is:

$$V^{(0)} = -\tan(\omega t) + C_1.$$

For the first approximation, we get:

$$V^{(1)} = -t + \tan(\omega t) - C_1 + C_2.$$

The total potential is obtained as:

$$V = V^{(0)} + V^{(1)} = -t + C_2. \quad (13)$$

The potential grows as time increases. The

$$V_{ij} = \begin{bmatrix} V(t)^{(1)} & \frac{W_1}{\cos(\omega t)}e^{i\omega t} \\ \frac{\bar{W}_1}{\cos(\omega t)}e^{-i\omega t} & V(t)^{(1)} \end{bmatrix}$$

this matrix is Hermitian, finally the matrix for order "*n*" appears as radial solution for Laplace equation, etc.

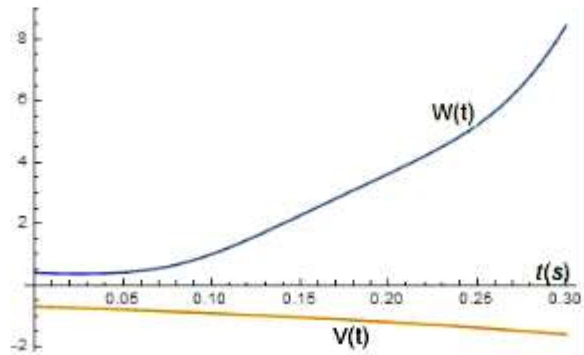


Figure 1: W(t) and V(t) versus t, V(t) extremely linearly decreases and W(t) exponentially increases.

### 3.2 Bessel's Quantum Amplitude

Secondly, we apply the above method to Bessel's quantum amplitude, such amplitudes appear in cylindrical quantum cavities. The general ODE form for Bessel's equation is:

$$a''(t) + a'(t)\frac{1}{t} + a(t)(1 - \frac{n^2}{t^2}) = 0. \tag{14}$$

By comparing equation (14) with equation (4), we get the following equations:

$$2V + \frac{W_1'W_1 + W_2'W_2}{|W|^2} = \frac{-1}{t}, \tag{15}$$

$$\omega|W|^2 = W_1W_2' - W_1'W_2, \tag{16}$$

$$V\left(\frac{W_1'W_1 + W_2'W_2}{|W|^2}\right) - |W|^2 + V^2 - V' = (1 - \frac{n^2}{t^2}). \tag{17}$$

We calculate the potential W using the previous section assumption as follows:

$$W = \frac{W_1}{\cos(\omega t)} e^{i\omega t}$$

and by using this assumption, we have:

$$|W|^2 = W_1^2 + W_2^2 = \frac{W_1^2}{\cos^2(\omega t)},$$

for the sake of simplicity, we take $|W|^2 = A^2$ , then we obtain the $W_1$ as:

$$W_1 = A\cos(\omega t). \tag{18}$$

By substituting equation (15) in equation (17) we obtain:

$$V' + V^2 + \frac{1}{t}V - \frac{n^2}{t^2} + A^2 + 1 = 0. \tag{19}$$

This is Riccati equation and can be integrated analytically, we obtain:

$$V = \frac{-((-\frac{1}{2}i\sqrt{-1-A^2}(Y_{n-1}(-i\sqrt{-1-A^2}t) - Y_{n+1}(-i\sqrt{-1-A^2}t))}{-Y_n(-i\sqrt{-1-A^2}t) - J_n(-i\sqrt{-1-A^2}t)C_1} - \frac{\frac{1}{2}i\sqrt{-1-A^2}(J_{n-1}(-i\sqrt{-1-A^2}t) - J_{n+1}(-i\sqrt{-1-A^2}t)))}{-Y_n(-i\sqrt{-1-A^2}t) - J_n(-i\sqrt{-1-A^2}t)C_1}. \tag{20}$$

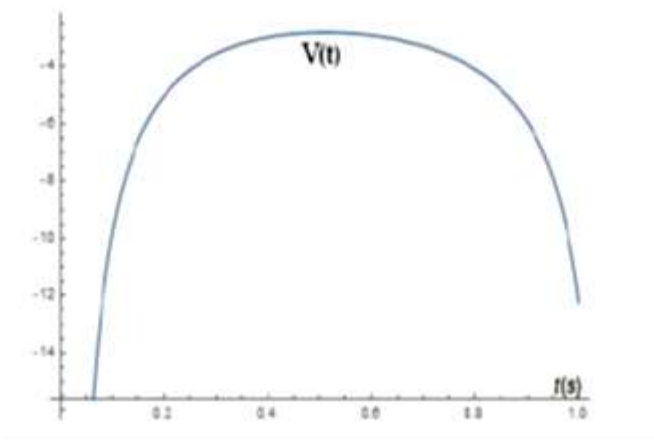


Figure 2: V for Bessel's is negative, it increases until it reaches approximately -5 and it remains constant for a period of time, after that it decreases.

### 3.3 Hermit Quantum Amplitude

The third example is Hermit quantum amplitude. It can be appeared as a solution to the harmonic oscillator in quantum mechanics, the general ODE form for Hermit equation is:

$$a''(t) - a'(t)2t + a(t)2n = 0. \tag{21}$$

We can find a set of differential equations just by following the same methodology:

$$2V + \frac{W_1'W_1 + W_2'W_2}{|W|^2} = 2t, \tag{22}$$

$$\omega|W|^2 = W_1W_2' - W_1'W_2, \tag{23}$$

$$V\left(\frac{W_1'W_1 + W_2'W_2}{|W|^2}\right) - |W|^2 + V^2 - V' = 2n. \tag{24}$$

We obtain a Riccati equation by substituting equation (22) in equation (24) as:

$$V' + V^2 - 2tV + 2n + A^2 = 0. \tag{25}$$

We get the potential V by solving the equation (25) as:

$$V = \frac{-(((2A+2n)C_1 H_{-1+\frac{1}{2}(2A+2n)}(t) + (-2A-2n)t_1F_1[1+\frac{1}{4}(-2A-2n);\frac{3}{2};2t])}{(-C_1 H_{\frac{1}{2}(2A+2n)}(t) - {}_1F_1(\frac{1}{4}(-2A-2n);\frac{1}{2};2t))}. \tag{26}$$

### 3.4 Laguerre Quantum Amplitude

The fourth example is Laguerre quantum amplitude which appears in Hydrogen atomic spectrum. The general ODE form for Laguerre function is:

$$a''(t) - a'(t)(\frac{1}{t} - 1) + a(t)\frac{n}{t} = 0. \tag{27}$$

By comparing equation (27) with equation (4), we get:

$$2V + \frac{W_1'W_1 + W_2'W_2}{|W|^2} = (1 - \frac{1}{t}), \tag{28}$$

$$\omega|W|^2 = W_1W_2' - W_1'W_2, \tag{29}$$

$$V\left(\frac{W_1'W_1 + W_2'W_2}{|W|^2}\right) - |W|^2 + V^2 - V' = \frac{n}{t}. \tag{30}$$

We obtain a Riccati equation by substituting equation (28) in equation (30) as:

$$V' + V^2 - (1 - \frac{1}{t})V + \frac{n}{t} + A^2 = 0. \tag{31}$$

By solving the equation (31), we get the potential V as:

$$V = \frac{C_1 * (f+g) + h}{k}. \tag{32}$$

where,

$$f = (\frac{1}{2} - \frac{1}{2}\sqrt{1-4A^2})e^{\frac{t}{2}-\frac{1}{2}\sqrt{1-4A^2}t}U(-\frac{1-\sqrt{1-4A^2}+2n}{2\sqrt{1-4A^2}}, 1, \sqrt{1-4A^2}t),$$

$$g=\frac{1}{2}e^{\frac{t}{2}-\frac{1}{2}\sqrt{1-4A^2}t}(1-\sqrt{1-4A^2}+2n)U(1-\frac{1-\sqrt{1-4A^2}+2n}{2\sqrt{1-4A^2}},2,\sqrt{1-4A^2}t),$$

$$h=(\frac{1}{2}-\frac{1}{2}\sqrt{1-4A^2})e^{\frac{t}{2}-\frac{1}{2}\sqrt{1-4A^2}t}L_{\frac{1-\sqrt{1-4A^2}+2n}{2\sqrt{1-4A^2}}}(\sqrt{1-4A^2}t)$$
$$-\sqrt{1-4A^2}e^{\frac{t}{2}-\frac{1}{2}\sqrt{1-4A^2}t}L^1_{-1+\frac{1-\sqrt{1-4A^2}+2n}{2\sqrt{1-4A^2}}}(\sqrt{1-4A^2}t)),$$

$$k=(-e^{\frac{t}{2}-\frac{1}{2}\sqrt{1-4A^2}t}C_1U(-\frac{1-\sqrt{1-4A^2}+2n}{2\sqrt{1-4A^2}},1,\sqrt{1-4A^2}t$$
$$-e^{\frac{t}{2}-\frac{1}{2}\sqrt{1-4A^2}t}L_{\frac{1-\sqrt{1-4A^2}+2n}{2\sqrt{1-4A^2}}}(\sqrt{1-4A^2}t)).$$

### 3.5 Modified Bessel's Amplitude

The last example is Modified Bessel's amplitude. The general ODE form for Modified Bessel's equation is:

$$a''(t)+a'(t)\left(\frac{1}{t}\right)-a(t)\left(1+\frac{n^2}{t^2}\right)=0. \tag{33}$$

We obtain the following equation by comparing equation (33) with equation (4):

$$2V+\frac{W_1'W_1+W_2'W_2}{|W|^2}=\left(-\frac{1}{t}\right), \tag{34}$$

$$\omega|W|^2=W_1W_2'-W_1'W_2, \tag{35}$$

$$V\left(\frac{W_1'W_1+W_2'W_2}{|W|^2}\right)-|W|^2+V^2-V'=-\left(1+\frac{n^2}{t^2}\right). \tag{36}$$

By substituting equation (34) in equation (36), we obtain a Riccati equation as:

$$V'+V^2-\frac{1}{t}V+A^2-\frac{n^2}{t^2}-1=0. \tag{37}$$

By solving the equation (37), we get the potential V as:

$$V=-\frac{-\frac{1}{2}i\sqrt{1-A^2}(Y_{n-1}(-i\sqrt{-1-A^2}t)-Y_{n+1}(-i\sqrt{-1-A^2}t))}{-Y_n(-i\sqrt{-1-A^2}t)-J_n(-i\sqrt{-1-A^2}t)C_1}-\frac{\frac{-1}{2}i\sqrt{1-A^2}(J_{n-1}(-i\sqrt{-1-A^2}t)-Y_{n+1}(-i\sqrt{-1-A^2}t))C_1}{-Y_n(-i\sqrt{-1-A^2}t)-J_n(-i\sqrt{-1-A^2}t)C_1}. \tag{38}$$

## 4. Distance Between Two Nearby Quantum States in Hilbert Space for Two Level System

In section 3, we used the amplitude to calculate the potential but, in this section, we will use the amplitude to obtain the distance between two nearby quantum states for two level system. This work is a non-relativistic version of the work done by Takayanagi in CFT [15]. If we have a perturbation parameter $\lambda$ (for example a very small external magnetic field), the $\psi$ will be function of $\lambda$ up to any order. All those $\psi(\lambda)$ are living at the Hilbert space and over time we have a little change in parameter, this change in the $\lambda$ leads to the quantum transition transition from $\psi(\lambda)$ to $\psi(\lambda+\delta\lambda)$. Here we want to measure the distance between $\psi(\lambda)$ and $\psi(\lambda+\delta\lambda)$ and how close they are to each other.

The higher correction formula of perturbation theory for time dependent potential is:

$$a_m(t) = a_m^{(0)} + \lambda a_m^{(1)} + \lambda^2 a_m^{(2)} + \ldots,$$

$$at \;\; m = 1\,, \quad a_1(t) = \delta_{1n} + \lambda a_1^{(1)} + \lambda^2 a_1^{(2)} + \ldots, \tag{39}$$

$$at \;\; m = 2\,, \quad a_2(t) = \delta_{2n} + \lambda a_2^{(1)} + \lambda^2 a_2^{(2)} + \ldots, \tag{40}$$

We assume that at t=0, the system at n=1 therefore prepared $\delta_{1n} = 1$ and $\delta_{2n} = 0$. The wavefunction of the ground state for two level system is:

$$\psi_1(\lambda) = (A + B\lambda)\phi_1^{(0)} + (\lambda C)\phi_2^{(0)}, \tag{41}$$

where

$$A = e^{\frac{-iE_1 t}{\hbar}}, B = e^{\frac{-iE_1 t}{\hbar}} \int_0^t V_{12}(t')e^{-i\omega_{12}t'}dt'$$

and

$$C = e^{\frac{-iE_2 t}{\hbar}} \int_0^t V_{21}(t')e^{-i\omega_{21}t'}dt'.$$

We define an auxiliary function

$$f_{12}(t,\omega_{12}) = \int_0^t V_{12}(t')e^{-i\omega_{12}t'}dt'$$

and

$$f_{21}(t,\omega_{21}) = \int_0^t V_{21}(t')e^{-i\omega_{21}t'}dt'.$$

Distance between two nearby quantum states in Hilbert space can be obtained by the inner product quantity as follows:

$$\begin{aligned} <\psi_1(\lambda)|\psi_1(\lambda+\delta\lambda)> &= <\psi_1(\lambda)|\psi_1(\lambda)> + \delta\lambda <\psi_1(\lambda)|B\phi_1^{(0)} + C\phi_2^{(0)}> \\ &= |A|^2 + \lambda^2|B|^2 + (A^*B + AB^*) + \lambda^2|C|^2 + \delta\lambda(BA^* + \lambda^2|B| + \lambda|C|^2). \end{aligned} \tag{42}$$

The result of inner product will contain real part and imaginary part as:

$$< \psi_1(\lambda)|\psi_1(\lambda+\delta\lambda) >= Re + iIm,$$

where,

$$Re = 1 + \frac{\lambda^2|f_{12}|^2}{\hbar^2} - \frac{2\lambda f_{12}^s(t,\omega_{12})}{\hbar} + \frac{\lambda^2|f_{21}(t,\omega_{21})|^2}{\hbar^2} + \delta\lambda\left(\frac{\lambda^2|f_{12}|^2}{\hbar^2} + \frac{\lambda|f_{21}(t,\omega_{21})|^2}{\hbar^2}\right),$$

$$Im = \delta\lambda\left(-\frac{f_{12}(t,\omega_{12})}{\hbar}\right).$$

By squaring both sides in (42) and dividing it by $a^2$,we obtain the final expression as:

$$\frac{|< \psi_1(\lambda)|\psi_1(\lambda+\delta\lambda) >|^2}{a^2} = 1 + \frac{2\left(\frac{\lambda^2|f_{12}|^2}{\hbar^2} + \frac{\lambda|f_{21}(t,\omega_{21})|^2}{\hbar^2}\right)}{1 + \frac{\lambda^2|f_{12}|^2}{\hbar^2} - \frac{2\lambda f_{12}^s(t,\omega_{12})}{\hbar} + \frac{\lambda^2|f_{21}(t,\omega_{21})|^2}{\hbar^2}}\delta\lambda +$$

$$\frac{\left(\frac{2\lambda^3|f_{12}|^2|f_{21}(t,\omega_{21})|\ \ \lambda^4|f_{12}|^4 + \lambda^2|f_{21}(t,\omega_{21})|^4}{\hbar^4} + \frac{|f_{12}(t,\omega_{12})|^2}{\hbar^2}\right)}{1 + \frac{\lambda^2|f_{12}|^2}{\hbar^2} - \frac{2\lambda f_{12}^s(t,\omega_{12})}{\hbar} + \frac{\lambda^2|f_{21}(t,\omega_{21})|^2}{\hbar^2}}\delta\lambda^2.$$

where,

$$\left(1 + \frac{\lambda^2|f_{12}|^2}{\hbar^2} - \frac{2\lambda f_{12}^s(t,\omega_{12})}{\hbar} + \frac{\lambda^2|f_{21}(t,\omega_{21})|^2}{\hbar^2} + \delta\lambda\left(\frac{\lambda^2|f_{12}|^2}{\hbar^2} + \frac{\lambda|f_{21}(t,\omega_{21})|^2}{\hbar^2}\right)\right)^2 \equiv a^2.$$

The distance between the $\psi_1(\lambda)$ and $\psi_1(\lambda+\delta\lambda)$ is:

$$\frac{|< \psi_1(\lambda)|\psi_1(\lambda+\delta\lambda) >|}{a} = \sqrt{\begin{array}{c} 1 + \frac{2\left(\frac{\lambda^2|f_{12}|^2}{\hbar^2} + \frac{\lambda|f_{21}(t,\omega_{21})|^2}{\hbar^2}\right)}{1 + \frac{\lambda^2|f_{12}|^2}{\hbar^2} - \frac{2\lambda f_{12}^s(t,\omega_{12})}{\hbar} + \frac{\lambda^2|f_{21}(t,\omega_{21})|^2}{\hbar^2}}\delta\lambda \\ + \frac{\left(\frac{2\lambda^3|f_{12}|^2|f_{21}(t,\omega_{21})|\ \ \lambda^4|f_{12}|^4 + \lambda^2|f_{21}(t,\omega_{21})|^4}{\hbar^4} + \frac{|f_{12}(t,\omega_{12})|^2}{\hbar^2}\right)}{1 + \frac{\lambda^2|f_{12}|^2}{\hbar^2} - \frac{2\lambda f_{12}^s(t,\omega_{12})}{\hbar} + \frac{\lambda^2|f_{21}(t,\omega_{21})|^2}{\hbar^2}}\delta\lambda^2 \end{array}}. \quad (43)$$

Using Taylor's series expansion, we obtain:

$$\frac{|<\psi_1(\lambda)|\psi_1(\lambda+\delta\lambda)>|}{a} = 1+\frac{1}{2}a\delta\lambda+[\frac{b}{2}-\frac{a^2}{8}]\delta\lambda^2+\ldots = 1+\frac{1}{2}a\delta\lambda+\chi_F\delta\lambda^2,$$

we define the fidelity susceptibility $\chi_F$ as a coefficient to the $(\delta\lambda^2)$ term:

$$\chi_F = \frac{\left(\frac{2\lambda^3|f_{12}|^2|f_{21}(t,\omega_{21})| \ \lambda^4|f_{12}|^4+\lambda^2|f_{21}(t,\omega_{21})|^4}{\hbar^4}+\frac{|f_{12}(t,\omega_{12})|^2}{\hbar^2}\right)}{2\left(1+\frac{\lambda^2|f_{12}|^2}{\hbar^2}-\frac{2\lambda f_{12}^s(t,\omega_{12})}{\hbar}+\frac{\lambda^2|f_{21}(t,\omega_{21})|^2}{\hbar^2}\right)^2} - \frac{\left(\frac{\lambda^2|f_{12}|^2}{\hbar^2}+\frac{\lambda|f_{21}(t,\omega_{21})|^2}{\hbar^2}\right)^2}{2\left(1+\frac{\lambda^2|f_{12}|^2}{\hbar^2}-\frac{2\lambda f_{12}^s(t,\omega_{12})}{\hbar}+\frac{\lambda^2|f_{21}(t,\omega_{21})|^2}{\hbar^2}\right)^2}. \quad (44)$$

## 5. Distance Between Two Nearby Quantum States in Hilbert Space for Landau-Zener Two Level Model

Landau-Zener is a transition between two energy levels with time-dependent Hamiltonian, it happens when the system exposed to periodic frequency with large amplitude (parameter F) gives $\chi_F$ in equation (44). Those transitions cause a phase difference $\delta F$ which leads to constructive or destructive interference. Our work in this section is to obtain the phase difference for Landau-Zener model by measuring the distance between $\psi(F)$ and $\psi(F+\delta F)$ and how close they are to each other.
The wavefunction of the ground state for two level system is:

$$\psi_0 = \sum_{k=1}^{2} a_{k0}\psi_k^{(0)} = a_{00}\psi_0^{(0)} + a_{10}\psi_1^{(0)}. \quad (45)$$

For the coefficients, we have:

$$\text{for level } m, \quad i\hbar\frac{da_m}{dt} = F_{mn}e^{i(\omega_{mn}-\omega)t}a_n = F_{mn}e^{i\alpha t}a_n, \quad (46)$$

$$\text{for level } n, \quad i\hbar\frac{da_n}{dt} = F_{mn}^{*}e^{i\epsilon t}a_m, \quad (47)$$

where $e^{i\epsilon t}a_n = b_n$.

We substitute equation (45) in (46), we obtain the ordinary differential equation as:

$$\frac{d^2b_n}{dt^2} - i\epsilon\frac{db_n}{dt} + \frac{|F_{mn}|^2}{\hbar^2}b_n = 0. \quad (48)$$

If we suppose that $|F_{mn}|^2$ =constant, we can solve equation (47) and find:

$$b_n = b_+ e^{i\left(\frac{\epsilon}{2}+\sqrt{\frac{\epsilon^2}{4}+\frac{|F_{mn}|^2}{\hbar^2}}\right)t} + b_- e^{i\left(\frac{\epsilon}{2}-\sqrt{\frac{\epsilon^2}{4}+\frac{|F_{mn}|^2}{\hbar^2}}\right)t},$$

$$a_n = \left(b_+ e^{i\left(\frac{\epsilon}{2}+\sqrt{\frac{\epsilon^2}{4}+\frac{|F_{mn}|^2}{\hbar^2}}\right)t} + b_- e^{i\left(\frac{\epsilon}{2}-\sqrt{\frac{\epsilon^2}{4}+\frac{|F_{mn}|^2}{\hbar^2}}\right)t}\right)e^{-i\epsilon t}. \tag{49}$$

Substituting $a_{00} \equiv a_0$ and $a_{10} \equiv a_1$ in (44), we obtain:

$$\psi_0 = \left(-\frac{iF_{10}^*}{\hbar\Omega} e^{-i\frac{\epsilon}{2}t}\sin(\Omega t)\right)\psi_0^{(0)} + (e^{i\frac{\epsilon}{2}t}\cos(\Omega t)) - \frac{i\epsilon}{2\Omega} e^{-i\frac{\epsilon}{2}t}\sin(\Omega t))\psi_1^{(0)}. \tag{50}$$

where

$$\Omega = \sqrt{\frac{\epsilon^2}{4} + \frac{|F_{mn}|^2}{\hbar^2}}.$$

The distance between the $\psi(F)$ and $\psi(F+\delta F)$ is:

$$\frac{<\psi_0(F)|\psi_0(F+\delta F)>}{<\psi_0(F)|\psi_0(F)>} = \sqrt{1 + \frac{4H\sin(\epsilon t)}{(1-M)}\delta F + \frac{G^2\cos^2(\epsilon t) + H^2\sin^2(\epsilon t)}{(1-M)^2}\delta F^2}. \tag{51}$$

where,

$$H = \frac{-8\epsilon F^3 t\cos\left(\sqrt{\epsilon^2 + \frac{4F^2}{\hbar^2}}\, t\right) + 2\epsilon F\sqrt{\epsilon^2 + \frac{4F^2}{\hbar^2}}\,\hbar^2 \sin\left(\sqrt{\epsilon^2 + \frac{4F^2}{\hbar^2}}\, t\right)}{(4F^2 + \epsilon^2\hbar^2)^2},$$

$$G = \frac{-8\epsilon F^3 t - 2\epsilon^3 F\hbar^2 t + 2\epsilon^3 F\hbar^2 t\cos\left(\sqrt{\epsilon^2 + \frac{4F^2}{\hbar^2}}\, t\right) + 2\epsilon F\sqrt{\epsilon^2 + \frac{4F^2}{\hbar^2}}\,\hbar^2 \sin\left(\sqrt{\epsilon^2 + \frac{4F^2}{\hbar^2}}\, t\right)}{(4F^2 + \epsilon^2\hbar^2)^2},$$

$$M = \frac{\epsilon\sin(\epsilon t)\sin\left(\sqrt{\epsilon^2 + \frac{4F^2}{\hbar^2}}\, t\right)}{\sqrt{\epsilon^2 + \frac{4F^2}{\hbar^2}}\, t}.$$

We continue our manipulations:

$$\frac{<\psi_0(F)|\psi_0(F+\delta F)>}{<\psi_0(F)|\psi_0(F)>} = 1 + \frac{2H\sin(\epsilon t)}{(1-M)}\delta F + [\frac{G^2\cos^2(\epsilon t) + H^2\sin^2(\epsilon t)}{2(1-M)^2} - \frac{2H^2\sin^2(\epsilon t)}{(1-M)^2}]\delta F^2 + \ldots$$

Finally, we suggest the fidelity susceptibility $\chi_F$ as:

$$\chi_F = \frac{G^2\cos^2(\epsilon t) + H^2\sin^2(\epsilon t)}{2(1-M)^2} - \frac{2H^2\sin^2(\epsilon t)}{(1-M)^2}. \tag{52}$$

## 6. Conclusion

In this work, we studied time dependent perturbation theory in two level quantum system and its applications. The single ODE of quantum amplitude of two-level system was derived and based on that, we reconstruct potential function V(t) and W(t) from a given quantum amplitude. Moreover, we found the distance between two nearby quantum states in Hilbert space and the distance for Landau-Zener effect as a two-level system.


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## 8. Appendix

This appendix includes the potentials W for all models examples in section 3.
The potential W for Bessel's equation is obtained as:

$$W = \sqrt{D + E + F + G} \tag{53}$$

where:

$$D = \frac{-1+\frac{n^2}{t^2}+(i\sqrt{-1-A^2}t)-(Y_{n-1}(-i\sqrt{-1-A^2}t)-Y_{n+1}(-i\sqrt{-1-A^2}t)+(J_{n-1}(-i\sqrt{-1-A^2}t)-J_{n+1}(-i\sqrt{-1-A^2}t)C_1))}{2t(Y_n(-i\sqrt{-1-A^2}t))}$$

$$E = \frac{Y_n(-i\sqrt{-1-A^2}t)C_1)-(-1-A^2)(Y_{n-1}(-i\sqrt{-1-A^2}t)-Y_{n+1}(-i\sqrt{-1-A^2}t)+(J_{n-1}((-i\sqrt{-1-A^2}t)-J_{n+1}((-i\sqrt{-1-A^2})t)C_1)^2)}{4(Y_n(-i\sqrt{-1-A^2}t)+J_n(-i\sqrt{-1-A^2}t)C_1)^2}$$

$$F = \frac{\begin{array}{c}(i\sqrt{-1-A^2}(Y_{n-1}(i\sqrt{-1-A^2}t)-Y_{n+1}(-i\sqrt{-1-A^2})t)+(J_{n-1}(-i\sqrt{-1-A^2})t)-J_{n+1}(-i\sqrt{-1-A^2}t))C_1) \\ (\frac{-1}{2}i\sqrt{-1-A^2})(Y_{n-1}((-i\sqrt{-1-A^2})t)\end{array}}{(2(Y_n((-i\sqrt{-1-A^2})t)+J_n((-i\sqrt{-1-A^2})t)C_1))^2)} - \frac{Y_{n+1}((-i\sqrt{-1-A^2})t))-1/2(-i\sqrt{-1-A^2})(J_{n-1}((-i\sqrt{-1-A^2}))t)[-1+n,(-i\sqrt{-1-A^2})t]}{(2(Y_n((-i\sqrt{-1-A^2})t)+J_n((-i\sqrt{-1-A^2})t)C_1))^2)} - \frac{Y_{n+1}((-i\sqrt{-1-A^2})t))-1/2(-i\sqrt{-1-A^2})(J_{n-1}(-i\sqrt{-1-A^2})t)-J_{n+1}((-i\sqrt{-1-A^2})t))C_1)))}{(2(Y_n((-i\sqrt{-1-A^2})t)+J_n((-i\sqrt{-1-A^2})t)C_1))^2)}$$

$$G = \frac{((i\sqrt{-1-A^2})(\frac{-1}{2}(i\sqrt{-1-A^2})(Y_{n-2}(-i\sqrt{-1-A^2})t)-Y_n((-i\sqrt{-1-A^2})t))}{(2(Y_n(-(i\sqrt{-1-A^2})t)+J_n(-(i\sqrt{-1-A^2})t)C_1)))} + \frac{\frac{1}{2}(i\sqrt{-1-A^2})(Y_n((-i\sqrt{-1-A^2})t)-Y_{n+2}((-i\sqrt{-1-A^2})t))+(\frac{-1}{2}(i\sqrt{-1-A^2})(J_{n-2}((-i\sqrt{-1-A^2})t)}{(2(Y_n(-(i\sqrt{-1-A^2})t)+J_n(-(i\sqrt{-1-A^2})t)C_1)))} -$$

$$\frac{J_n(-(-i\sqrt{-1-A^2})t))+\frac{1}{2}((-i\sqrt{-1-A^2})(J_n((-i\sqrt{-1-A^2})t)-J_{n+2}((-i\sqrt{-1-A^2})t)))C_1)))}{(2(Y_n(-(i\sqrt{-1-A^2})t)+J_n(-(i\sqrt{-1-A^2})t)C_1)))}$$

The potential W for Hermit equation is obtained as:

$$W = A \tag{54}$$

The potential W for Laguerre equation is obtained as:

$$W = \sqrt{\frac{-(A^2(3(-4+t-\sqrt{1-4A^2}t+2nt)_1F_1(\frac{-1+3\sqrt{1-4A^2}-2n}{2\sqrt{1-4A^2}};3;\sqrt{1-4A^2})}{(12(C_1U(\frac{-1+3\sqrt{1-4A^2}-2n}{2\sqrt{1-4A^2}};3;\sqrt{1-4A^2}t)+L_{\frac{1}{2}(-1+\frac{1}{\sqrt{1-4A^2}}+\frac{2n}{\sqrt{1-4A^2}})}(\sqrt{1-4A^2})t}} + \sqrt{\frac{(1+3\sqrt{1-4A^2}+2n)t_1F_1(\frac{-1+3\sqrt{1-4A^2}-2n}{2\sqrt{1-4A^2}};4;\sqrt{1-4A^2})}{(12(C_1U(\frac{-1+3\sqrt{1-4A^2}-2n}{2\sqrt{1-4A^2}};3;\sqrt{1-4A^2}t)+L_{\frac{1}{2}(-1+\frac{1}{\sqrt{1-4A^2}}+\frac{2n}{\sqrt{1-4A^2}})}(\sqrt{1-4A^2})t}} - \sqrt{\frac{12(C_1U(\frac{-1+3\sqrt{1-4A^2}-2n}{2\sqrt{1-4A^2}};3;\sqrt{1-4A^2}t))}{(12(C_1U(\frac{-1+3\sqrt{1-4A^2}-2n}{2\sqrt{1-4A^2}};3;\sqrt{1-4A^2}t)+L_{\frac{1}{2}(-1+\frac{1}{\sqrt{1-4A^2}}+\frac{2n}{\sqrt{1-4A^2}})}(\sqrt{1-4A^2})t}} \tag{55}$$

The potential W for Modified Bessel's equation is obtained as:

$$W = \sqrt{\frac{-((8n-4A^2t^2)Y_{n-2}(-i\sqrt{-1-A^2}t)}{4t^2\left(Y_n\left(-i\sqrt{-1-A^2}t\right)+J_n\left(-i\sqrt{-1-A^2}t\right)C_1\right)}} + \sqrt{\frac{\frac{-8i(2(-1+n)n+(1-A^2n)t^2)Y_{n-1}\left(-i\sqrt{-1-A^2}t^2\right)-4\sqrt{1-A^2}t(-2n+A^2t^2)J_{n-2}\left(-i\sqrt{-1-A^2}t\right)C_1}{\sqrt{1-A^2}t}}{4t^2\left(Y_n\left(-i\sqrt{-1-A^2}t\right)+J_n\left(-i\sqrt{-1-A^2}t\right)C_1\right)}} + \sqrt{\frac{\frac{-8i(2(-1+n)n+(1-A^2n)t^2)J_{n-1}(-i\sqrt{-1-A^2}t^2)C_1}{\sqrt{1-A^2}t}}{4t^2(Y_n(-i\sqrt{-1-A^2}t)+J_{n-1}(-i\sqrt{-1-A^2}t)C_1)}} \tag{56}$$

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