Notification No.: 577/2025
Date of Award: 07-04-2025

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Topic of Research: "A Study of s-Numbers and Entropy Numbers of Linear Operators"

Findings

Linear operators play an important role in the study of a number of mathematical problems. But it is not always possible to explore the solutions of such problems by applying the standard methods. Such situations require the theoretical study of operators. The main aim of this thesis is to investigate and provide details about the following notions which nowadays are emerging rapidly for exploring fundamental properties of operators: degree of compactness, measure of non-compactness, finite strict singularity/co-singularity etc. In fact, we present the theory of **s-numbers** (introduced by A. Pietsch [3]) and **entropy numbers** of operators as important tools for exploring such fundamental properties. Also, we investigate the above properties for some well-known operators by estimating their s-numbers and entropy numbers.

A number of articles have been published which have studied the problem of coincidence of strict s-numbers of operators. In case of Hilbert spaces, the coincidence of strict s-numbers is well-known. The equality of strict s-numbers of certain well-known operators and embeddings acting between Banach spaces has been proved in [1, 2]. We have proved the equality of strict s-numbers of weighted Hardy-type operators acting between Lebesgue spaces over a tree contained in Chapter 2.
This thesis comprises of five chapters.

Chapter 1 presents definitions of various Banach function spaces. Standard spaces: Sobolev, $B^s_{p,q}$ and $F^s_{p,q}$ and their embeddings, useful in the theoretical study of partial differential equations, are introduced. Theory of s-numbers, entropy numbers and their applications to explore fundamental properties such as degree of compactness and measure of non-compactness are embodied.

Chapter 2 studies strict s-numbers of weighted Hardy-type operators on trees. Construction of Lebesgue spaces over a tree is given and Hardy-type operators are defined thereon. Coincidence of strict s-numbers of these operators has been established.

Chapter 3 embodies theoretical study of Sobolev embeddings $E : W_0^{2,1}(a,b) \hookrightarrow C[a,b]$ and $E_0 : W_{00}^{2,1}(a,b) \hookrightarrow C[a,b]$. Bernstein and isomorphism numbers are calculated and finite strict singularity for these embeddings is studied. Results obtained are compared with the results of [4].

Chapter 4 contains study of non-compact Volterra operators $V_a : L_1[0, 1 + a] \to C[0, 1], 0 \le a < \infty$, defined by $(V_a f)(x) = \int_0^{(1+a)x} f(t)dt$. Exact values of all strict s-numbers of V_a are obtained and through Bernstein and Mityagin numbers, finite strict singularity and finite strict co-singularity are investigated.

Chapter 5 is devoted to study Laplace transform operator $\mathcal{L} : L^1(0, \infty) \to L^\infty(0, \infty)$ defined by $\mathcal{L}f(x) := \displaystyle\int_0^\infty f(t)e^{-xt}dt,\ x \in (0, \infty)$ by restricting to the real case. Entropy numbers of this operator are estimated and Measure of non-compactness is investigated. Strict s-numbers of $\mathcal{L}$ are determined and its finite strict singularity/ co-singularity are examined by studying Bernstein and Mityagin numbers, respectively. It is observed that estimates of isomorphism numbers involve certain type of Hilbert matrices.

References

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